DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

September 1998

No aids.
Do all questions.
All questions are of equal value.

1. Suppose \( A \) is an \( 8 \times 8 \) matrix whose characteristic polynomial is \( \lambda^3(\lambda - 1)^2(\lambda - 5)^3 \) and whose minimal polynomial is \( \lambda(\lambda - 1)^2(\lambda - 5)^2 \). What is the dimension of each of its eigenspaces?

2. Classify all groups of order 12.

3. What is the Galois group of (the splitting field of) \( X^3 - 2 \) over \( \mathbb{Q} \)? (Explain your answer.)

4. Give a set of representatives for the conjugacy classes in the symmetric group \( S_5 \), and find the number of elements in each class.

5. If \( p \) is a prime, what is the order of \( GL(2, \mathbb{F}_p) \), the group of invertible \( 2 \times 2 \) matrices over the field with \( p \) elements?

6. For which primes \( p \) is every element of the finite field \( \mathbb{F}_p \) equal to the fifth power of an element of the field?

7. Suppose \( f \in \mathbb{Q}[x] \) is an irreducible cubic polynomial for which the Galois group of the splitting field of \( f \) over \( \mathbb{Q} \) is not abelian. How many subfields does the splitting field of \( f \) have, and how many of them are normal? (Explain your answer.)
8. (a) What are the maximal ideals in $\mathbb{C}[x]$? Explain your answer.

(b) Give an example of a ring that contains a prime ideal that is not maximal. Explain your answer.

9. What are the maximal ideals in $\mathbb{R}[x]$? Explain your answer. (Hint: What is $\mathbb{R}[x]/(x^2 + 1)$, where $(x^2 + 1)$ means the principal ideal generated by $x^2 + 1$?)

10. Both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are modules over the ring $\mathbb{Z}_6$ in a natural way. Identify the tensor product $\mathbb{Z}_2 \otimes_{\mathbb{Z}_6} \mathbb{Z}_3$.

(Here $\mathbb{Z}_m$ means $\mathbb{Z}/(m\mathbb{Z})$...