DEPARTMENT OF MATHEMATICS
University of Toronto

Algebra Exam (3 hours)

January 1997

No aids.
Do all questions.

1. [20 points]
   (a) Prove that the following conditions on a ring $R$ are equivalent.
   (i) $R$ satisfies the ascending chain condition for left ideals.
   (ii) Any nonempty set $\mathcal{S}$ of left ideals has a maximal element.
   (iii) Any left ideal in $R$ is finitely generated.
   (b) Suppose that $R$ satisfies the three conditions of (a), and that $I$ is a left ideal in $R[x]$. Show that any $R[x]$-submodule of $R[x]/I$ is finitely generated.

2. [15 points]
   (a) Prove that any element $g \in GL(n, \mathbb{C})$ has an eigenvalue.
   (b) Let $G_{1,2}$ be the subset of $G = GL(4, \mathbb{C})$ whose eigenvalues are in the set $\{1, 2\}$. Prove that $G_{1,2}$ is invariant under conjugation by $G$.
   (c) How many $G$-conjugacy classes are there in $G_{1,2}$?

3. [15 points]
   Suppose that $R$ is an integral domain.
   (a) Define a prime element and an irreducible element in $R$.
   (b) Prove that any prime element is irreducible.
   (c) Prove that any irreducible element $f(x) \in \mathbb{Z}[x]$ is prime.
   (d) Write down an irreducible polynomial in $\mathbb{Z}[x]$ of degree 6. (Explain your reasons.)
4. [30 points]
   a) State Sylow’s theorem.
   b) What is the order of the group

   \[ SL(2, \mathbb{F}_3) = \{ g \in M_2(\mathbb{F}_3) : \det(g) = 1 \}, \]

   and what is the order of its center \( Z \)?
   c) Write down a Sylow 3-subgroup of \( SL(2, \mathbb{F}_3) \).
   d) How many Sylow 3-subgroups does \( SL(2, \mathbb{F}_3) \) have?
   e) Show that \( SL(2, \mathbb{F}_3)/Z \cong A_4 \).

5. [20 points]
   Let \( f(x) \in F[x] \) be a monic polynomial of degree \( n \).
   a) What is meant by the splitting field \( E \) of \( f(x) \) over \( F \)?
   b) Define the Galois group of \( E/F \), and the Galois group of \( f(x) \) over \( F \), and describe
      how the two are related.
   c) If \( \{ \alpha_1, \ldots, \alpha_n \} \) are the roots of \( f(x) \), show that

   \[ D = \prod_{1 \leq i, j \leq n} (\alpha_i - \alpha_j)^2 \]

   belongs to \( F \).
   d) Show that the Galois group of \( f(x) \) over \( F \) is contained in \( A_n \) if and only if \( D \)
      has a square root in \( F \).