MAT247S, 2009 Winter, Problem Set 5 Solution

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1. By finite dimensionality, one can write \( W = \bigoplus_{i=0}^{n} F(T^n(x)) \) for some finite \( n \). Then (a) and (b) are straightforward.

2. As hinted, write \( V = \bigoplus_{i=0}^{n} F(T^n(x)) \) and \( U(x) = g(T)(x) \) as in Ex 1, then for \( k = 0, \ldots, n \) we have \( UT^k(x) = T^kU(x) \) (since \( UT = TU \)) = \( T^kg(T)(x) = g(T)T^k(x) \) (since \( T \) and \( g(T) \) clearly commute). We have shown \( U = g(T) \) on the basis \( x, T(x), \ldots, T^n(x) \), hence they are equal.

3. (8 marks) Let \( W \neq 0 \) be a \( T \)-invariant subspace of \( V \). Let \( m_T \) and \( m_{T_W} \) be the minimal polynomial for \( T \) and \( T_W \) respectively. Then we have \( m_{T_W} \) divides \( m_T \) since \( m_T(T_W) = (m_T(T))_W = 0 \). Also the characteristic polynomial of \( T \) splits, so does its factor \( m_T \), and also \( m_{T_W} \). As \( W \neq 0 \) we know \( \deg(m_{T_W}) \geq 1 \), which means \( m_{T_W} \) has at least a linear factor \( t - \lambda \). So \( \ker(T - \lambda I) \) (or \( N(T - \lambda I) \) in the book) is non zero by definition of \( m_{T_W} \) being minimal. Hence \( 0 \neq v \in \ker(T - \lambda I) \) is an eigenvector. (If \( m_{T_W} \) does not split, there may not exist any eigenvector for \( T_W \)!!)

2 marks for showing \( m_{T_W} \) divides \( m_T \).
2 for showing \( m_{T_W} \) splits
2 for showing \( W \neq 0 \Rightarrow \deg(m_{T_W}) \geq 1 \). (Most of you did not mention this.)
2 for showing eigenvalue and eigenvector exists.

4. (7 marks total) Let the minimal polynomial of \( T \) be \( m_T(t) = \sum_{i=0}^{d} a_i t^i \) with \( a_d = 1 \). So we have \( m_T(T) = \sum_{i=0}^{d} a_i T^i = T_0 \), in particular \( \sum_{i=0}^{d} a_i T^i(x) = 0 \). So \( T^d(x) = -\sum_{i=0}^{d-1} a_i T^i(x) \) (2 marks). Using this iteration we have \( W = \text{span}\{x, T(x), \ldots, T^{d-1}(x)\} \) (2 marks), i.e. \( W \) is spanned by \( d \) vectors, hence \( \dim(W) \leq d \) (3 marks).

5. (10 marks) (Here I only check \( j = 3 \). The method also applies for \( j = 1, 2 \) and indeed for any \( j \) as long as we have eigenvectors of \( j \) distinct eigenvalues.)
First show \( \{y_3, T(y_3), T^2(y_3)\} \) are linearly independent. Suppose \( u y_3 + v T(y_3) + w T^2(y_3) = 0 \) for some \( u, v, w \in F \). Now writing \( T^k(y_3) = T^k(x_1 + x_2 + x_3) = \lambda_1^k x_1 + \lambda_2^k x_2 + \lambda_3^k x_3 \), we have
\[
(1) \quad (u + v \lambda_1 + w \lambda_1^2)x_1 + (u + v \lambda_2 + w \lambda_2^2)x_2 + (u + v \lambda_3 + w \lambda_3^2)x_3 = 0.
\]
Recall eigenvectors corresponding to different eigenvalues are linearly independent. Hence above equation (1) gives
\[
\begin{pmatrix}
1 & \lambda_1 & \lambda_1^2 \\
1 & \lambda_2 & \lambda_2^2 \\
1 & \lambda_3 & \lambda_3^2
\end{pmatrix}
\begin{pmatrix}
u \\
w
1
\end{pmatrix}
= \begin{pmatrix}0 \\
0 \\
0
\end{pmatrix}.
\]
One check the matrix above has determinant $(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \neq 0$ as all $\lambda_i$ distinct. The only solution for $(u, v, w)$ is $(0, 0, 0)$. We have shown \{$y_3, T(y_3), T^2(y_3)$\} linearly independent.

To claim $\dim(W_3) = 3$, it suffices to show any $T^k(y_3)$ is a linear combination of \{$y_3, T(y_3), T^2(y_3)$\} for $k \geq 3$, i.e. there exists $u, v, w$ so that

$$u y_3 + v T(y_3) + w T^2(y_3) = T^k(y_3).$$

Again writing (3) in terms (4)

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \lambda_1^k \\ \lambda_2^k \\ \lambda_3^k \end{pmatrix}.$$

Again since the determinant of the matrix is non zero, we can solve for $(u, v, w)$ which cannot be all 0.

2 marks for setting up equations (2), (4).
3 for showing the determinant is non-zero.
5 for showing \{$y_3, T(y_3), T^2(y_3)$\} being a basis, i.e. linearly independent and spanning.

6. Let $W_k = \text{span}_F\{I_V, T, \ldots, T^k\}$. Then $W_k$ is an increasing sequence of $F$-subspace of $W$. Since $V$ is, and so are $L(V)$ and $W$, finite dimensional, there is $k$ so that $W_k = W_{k+1} = \cdots = W$. By taking minimal such $k$ we have $\dim(W) = k + 1$. Now $W_k = W_{k+1}$ means $T^{k+1} = \sum_{j=0}^k a_j T^j$ for some $a_j \in F$. This is a polynomial equation for $T$, so its minimal polynomial $m_T(x)$ must have degree $d \leq k + 1 = \dim(W)$. Now if $d \leq k + 1$, then it is easily seen that $W_{d-1} = W_d = \cdots$, contradicting the minimality of $k$. So $d = k + 1 = \dim(W)$.

7. (a) One can show $(p(T))^* = \overline{p}(T^*)$, and using $\langle v, p(T)^*(w) \rangle = \langle p(T)(v), w \rangle = \langle 0, w \rangle = 0$ for any $v, w \in V$, we have $\overline{p}(T^*) = 0$.
(b) Since $\overline{p}(T^*) = 0$, we have the minimal polynomial $m_{T^*}$ divides $\overline{p}$. Now if $\deg(m_{T^*}) \leq \deg(\overline{p})$, then notice $m_{T^*}(T) = \overline{m_{T^*}(T^*)} = (m_{T^*}(T^*))^* = 0^* = 0$, we have $T$ satisfies the polynomial $\overline{m_{T^*}}$ whose degree is $\leq \deg(p)$, contradiction. So $\deg(m_{T^*}) = \deg(\overline{p})$ and hence $m_{T^*} = \overline{p}$ as both are monic.

8. If there is $g = \sum_{i=0}^n a_i t^i \in P(\mathbb{R})$ that $g(T) = T_0$, then all polynomials in $V = P(\mathbb{R})$ satisfy a differential equation: $\sum_{i=0}^n a_i f^{(i)} = 0$ for all $f \in V$. This is absurd, for example we can take $f(t) = t^n$, then it is easily checked that $g(T)(f) = n! a_n \neq 0$ (where we can assume $a_n \neq 0$ without loss of generality).

9. (a) Since $P^{-1} T P = D$ implies $P^{-1} T^2 P = D^2$.
(b) (5 marks) The fact $g(T^2) = 0$ implies $T$ satisfies $g(t^2) = 0$, hence $p(t)$, as the minimal polynomial of $T$, must divide $g(t^2)$.
(c) (8 marks total) Given $T$ is diagonalizable, suppose $\{\lambda_1, \ldots, \lambda_N\}$ are distinct eigenvalues for $T$, then

(2 marks) the minimal polynomial $p(t) = \prod_{i=1}^{N} (t - \lambda_i)$.

(2 marks) Given $p(t) = p(-t)$, we have $\lambda$ is an eigenvalue iff $-\lambda$ is. 

(2 marks) We can pair the roots by sign because all $\lambda_i \neq 0$ (given $T$ invertible).

Hence $N$ is even $2n$ and we have $p(t) = \prod_{i=1}^{n} (t^2 - \lambda_i^2)$.

(2 marks) Now $T^2$ is also diagonalizable, and its eigenvalues are $\{\lambda_i^2|i = 1, \ldots, n\}$ all distinct, so its minimal polynomial is $g(t) = \prod_{i=1}^{n} (t - \lambda_i^2)$. Hence $p(t) = g(t^2)$.

1 marks will be deducted if you write $T = \text{diag}(\lambda_1, \ldots, \lambda_m)$ implies $p(t) = \prod_{i=1}^{m} (t - \lambda_i)$. It is because $T$ may have equal eigenvalues.

10.(a) Let $m$ be the minimal polynomial for $T$, then $m(T) = 0$ implies (for $i = 1, 2$) $m(T_{W_i}) = (m(T))_{W_i} = 0$, i.e. $T_{W_i}$ satisfies $m(t) = 0$. Hence $p_i$ divides $m$.

(b)(c) Answer is open. For example, one may take $T$ to be diagonal with equal eigenvalue in (b) and different eigenvalues in (c), and $W_i$ be (sum of) different eigenspaces.

7 marks total for (b). 1 each of $V, W_1, W_2, T, p, p_1$ and $p_2$.

11.(a) Choose a basis for $W_1$ and extend to one of $V$, then $[T]$ is block upper triangular, i.e. of the form \( \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \), then $g_1$ is the minimal polynomial of $D$.

(b) If $h(T)(V) \subseteq W_1$, then $h(D) = 0$, so $g_1$ divides $h$ as minimal.

(c) Since $0 = m_T([T]) = \begin{pmatrix} m_T(A) & * \\ 0 & m_T(D) \end{pmatrix}$, we have $m_T(D) = 0$ and so $g_1$ divides $m_T$. Also because $m_T$ divides $ch_T$, we have $g_1$ divides $ch_T$.

(d) Choose a basis of $W_2$, extend to one of $W_1$, and further to one of $V$. Then $A$ and $B$ are further divided, say $A = \begin{pmatrix} E & F \\ 0 & J \end{pmatrix}$, $B = \begin{pmatrix} J_K \\ 0 \end{pmatrix}$. We have $[T] = \begin{pmatrix} E & F & J \\ 0 & F & J_K \\ 0 & 0 & D \end{pmatrix}$. Take $g_2$ to be the minimal polynomial of the $2 \times 2$ lower diagonal block $\begin{pmatrix} J_K \\ 0 \end{pmatrix}$. 