5. Let $V$ be a finite-dimensional real inner product space and let $m$ be a positive integer. Suppose that $T \in \mathcal{L}(V)$, $T$ is normal, and $T^m = T_0$ (that is, $T^m(x) = 0$ for all $x \in V$). Prove that $T = T_0$.

Solution: Because $V$ is a real inner product space and we only assumed that $T$ is normal (not necessarily self-adjoint), we cannot assume that there exists an orthonormal basis for $V$ consisting of eigenvectors of $T$. Let $\beta = \{x_1, \ldots, x_n\}$ be an orthonormal basis for $V$. Let $A = [T]_\beta$. Because $\beta$ is orthonormal, we know that $A^* = [T^*]_\beta$. Because $TT^* = T^*T$, we have that

$$AA^* = [T]_\beta[T^*]_\beta = [TT^*]_\beta = [T^*T]_\beta = [T^*]_\beta[T]_\beta = A^*A.$$

That is, $A$ is a normal matrix. Because the real numbers are a subset of the complex numbers, we can consider $A$ as a complex matrix. We still have $AA^* = A^*A$ because complex conjugation has no effect on real numbers. So $A$ is a normal complex matrix. Therefore there exists a unitary matrix $P$ and a diagonal matrix $D$ such that $A = P^{-1}DP$. Note that $[T^m]_\beta = ([T]_\beta)^m = A^m = 0$. It follows that

$$0 = A^m = (P^{-1}DP)^m = P^{-1}D^mP.$$

This implies $D^m = PAP^{-1} = 0$. Now $D$ is diagonal, so $D^m = 0$ implies $D = 0$. This tells us that $A = P^{-1}0P = 0$. Since $[T_0]_\beta = 0$, we have shown that $[T]_\beta = [T_0]_\beta$. Because the map $U \mapsto [U]_\beta$ from $\mathcal{L}(V)$ to $M_{n \times n}(\mathbb{R})$ is one-to-one, we conclude that $T = T_0$. 