The 13 Postulates

Everything you ever wanted to know about the real numbers is summarized as follows. There is a set \( \mathbb{R} \) “of real numbers” with two binary operations defined on it, + and \( \cdot \) (“addition” and “multiplication”), two different distinct elements 0 and 1 and a subset \( \mathbb{P} \) “of positive numbers” so that the following 13 postulates hold:

P1 Addition is associative: \( \forall a, b, c \ a + (b + c) = (a + b) + c \) (“\( \forall \)” means “for every”).

P2 The number 0 is an additive identity: \( \forall a \ a + 0 = 0 + a = a \).

P3 Additive inverses exist: \( \forall a \exists (-a) \text{ s.t. } a + (-a) = (-a) + a = 0 \) (“\( \exists \)” means “there is” or “there exists”).

P4 Addition is commutative: \( \forall a, b \ a + b = b + a \).

P5 Multiplication is associative: \( \forall a, b, c \ a \cdot (b \cdot c) = (a \cdot b) \cdot c \).

P6 The number 1 is a multiplicative identity: \( \forall a \ a \cdot 1 = 1 \cdot a = a \).

P7 Multiplicative inverses exist: \( \forall a \neq 0 \exists a^{-1} \text{ s.t. } a \cdot a^{-1} = a^{-1} \cdot a = 1 \).

P8 Multiplication is commutative: \( \forall a, b \ a \cdot b = b \cdot a \).

P9 The distributive law: \( \forall a, b, c \ a \cdot (b + c) = a \cdot b + a \cdot c \).

P10 The trichotomy for \( \mathbb{P} \): for every \( a \), exactly one of the following holds: \( a = 0 \), \( a \in \mathbb{P} \) or \( (-a) \in \mathbb{P} \).

P11 Closure under addition: if \( a \) and \( b \) are in \( P \), then so is \( a + b \).

P12 Closure under multiplication: if \( a \) and \( b \) are in \( P \), then so is \( a \cdot b \).

P13 The thirteenth postulate is the most subtle and interesting of all. It will await a few weeks.

Here are a few corollaries and extra points:

1. Sums such as \( a_1 + a_2 + a_3 + \cdots + a_n \) are well defined.
2. The additive identity is unique. (Also multiplicative).
3. Additive inverses are unique. (Also multiplicative).
4. Subtraction can be defined.
5. \( a \cdot b = a \cdot c \) iff (if and only if) \( a = 0 \) or \( b = c \).
6. \( a \cdot b = 0 \) iff \( a = 0 \) or \( b = 0 \).
7. \( x^2 - 3x + 2 = 0 \) iff \( x = 1 \) or \( x = 2 \).
8. \( a - b = b - a \) iff \( a = b \).
9. A “well behaved” order relation can be defined (i.e., the Boolean operations \( <, \le, > \) and \( < \) can be defined and they have all the expected properties).
10. The “absolute value” function \( a \mapsto |a| \) can be defined and for all numbers \( a \) and \( b \) we have \( |a + b| \le |a| + |b| \).