Problem 1. Is there a non-zero polynomial $p(x)$ defined on the interval $[0, \pi]$ and with values in the interval $[0, \frac{1}{2})$ so that it and all of its derivatives are integers at both the point $0$ and the point $\pi$? In either case, prove your answer in detail. (Hint: How did we prove the irrationality of $\pi$?)

Solution. There isn’t. Had there been one, we could reach a contradiction as in the proof of the irrationality of $\pi$. Indeed we would have that $0 < \int_0^\pi p(x) \sin x \, dx < \frac{1}{2} \int_0^\pi \sin x \, dx = 1$, hence the integral $I = \int_0^\pi p(x) \sin x \, dx$ is not an integer. But repeated integration by parts gives

$$I = \int_0^\pi p'(x) \cos x \, dx = \int_0^\pi p''(x) \sin x \, dx = \ldots$$

The assumptions on $p^{(k)}(0) \in \mathbb{Z}$ and $p^{(k)}(\pi) \in \mathbb{Z}$ along with the fact that $\sin 0$, $\sin \pi$, $\cos 0$ and $\cos \pi$ are all integers imply that the boundary terms are all integers. If $n$ is large enough, $p^{(2n)} = 0$ and hence the remaining integral is $0$. So $I$ is an integer, and that’s a contradiction.

Problem 2. Compute the volume $V$ of the “Black Pawn” on the right — the volume of the solid obtained by revolving the solutions of the inequalities $4x^2 \leq y + 3 - (y - 3)^3$ and $y \geq 0$ about the $y$ axis (its vertical axis of symmetry). (Check that $5 + 3 - (5 - 3)^3 = 0$ and hence the height of the pawn is $5$).

Solution. This is the area of the rotation solid with radius $r(y) = \frac{1}{2} \sqrt{y + 3 - (y - 3)^3}$ bounded by $y = 0$ and $y = 5$. Thus

$$V = \pi \int_0^5 r(y)^2 \, dy = \frac{\pi}{4} \int_0^5 (y + 3 - (y - 3)^3) \, dy$$

$$= \frac{\pi}{4} \left| \frac{y^2}{2} + 3y - \frac{(y - 3)^4}{4} \right|_0^5 = \frac{175\pi}{16}.$$
Solution.

1. $f' = \frac{1}{(1-x)^2}$, $f'' = \frac{2}{(1-x)^3}$, $f''' = \frac{2\cdot3}{(1-x)^4}$ and so it can be shown by induction that $f^{(k)} = \frac{k!}{(1-x)^{k+1}}$. Thus $f^{(k)}(0) = k!$ and hence $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$.

2. Cauchy’s formula for the remainder is $R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1} = \frac{x^{n+1}}{(1-t)^{n+2}}$ for some $t \in [0, x]$.

3. If $|x| < \frac{1}{2}$ then $|t| < \frac{1}{2}$ and $|1 - t| > \frac{1}{2}$ and hence $\left| \frac{x}{1-t} \right| < 2|x| < 1$ and thus $|R_n(x)| < \frac{1}{1-t}(2|x|)^{n+1} \to 0$. Therefore $f(x) - P_n(x) \to 0$, as required.

Problem 4.

1. Prove that if $\lim_{n \to \infty} a_n = l$ and the function $f$ is continuous at $l$, then $\lim_{n \to \infty} f(a_n) = f(l)$

2. Let $b > 1$ be a number, and define a sequence $a_n$ via the relations $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + b/a_n)$ for $n \geq 1$. Assuming that this sequence is convergent to a positive limit, determine what this limit is.

Solution.

1. See the “easy” part of Theorem 1 of Spivak’s Chapter 22.

2. Assume $\lim a_n = l > 0$. Then $l = \lim a_{n+1} = \lim \frac{1}{2}(a_n + b/a_n)$. Using the first part of this question on the function $x \mapsto \frac{1}{2}(x + b/x)$, which is continuous at $l$, we find that $\lim \frac{1}{2}(a_n + b/a_n) = \frac{1}{2}(l + b/l)$. Hence $l$ satisfies $l = \frac{1}{2}(l + b/l)$. Dividing by $l$ we get $1 = \frac{1}{2} + \frac{b}{2l}$ which is $1 = \frac{b}{2}$ which along with $l > 0$ implies that $l = \sqrt{b}$.

Problem 5. Do the following series converge? Explain briefly why or why not:

1. $\sum_{n=1}^{\infty} \frac{n}{2n+1}$.

   Solution. $\lim_{n \to \infty} \frac{n}{2n+1} = \frac{1}{2}$ hence by the vanishing test the series cannot converge.

2. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n\sqrt{n}+1}$.

   Solution. $\frac{\sqrt{n}}{n\sqrt{n}+1} > \frac{\sqrt{n}}{2n\sqrt{n}} = \frac{1}{2n}$. The latter is a multiple of the harmonic series which doesn’t converge, hence the original series doesn’t converge either.

3. $\sum_{n=1}^{\infty} \frac{n^2}{n!}$.

   Solution. Ignoring the first two terms of the series, which don’t change convergence anyway,
   \[
   \frac{n^2}{n!} = \frac{n^2}{n(n-1)(n-2)!} < \frac{n^2}{2n^2(n-2)!} = \frac{1}{2(n-2)!}.
   \]
The latter sequence is summable as we have shown in class, hence the original series is convergent.

4. \( \sum_{n=1}^{\infty} \frac{\log n}{n^2} \).

**Solution.** The function \( f(x) = \sqrt{x} - \log x \) is positive at \( x = 1 \) and simple differentiation shows that \( f'(x) > 0 \) for \( x \geq 1 \), hence it is increasing, and hence it is positive for all \( x \geq 1 \). Thus \( \frac{\log n}{n^2} < \frac{\sqrt{n}}{n^{3/2}} = \frac{1}{n^{3/2}} \) which is summable as was shown in class.

5. \( \sum_{n=2}^{\infty} \frac{1}{n \log n} \).

**Solution.** That’s a tough one. Here’s a solution inspired by the solution to Problem 20 of Spivak’s Chapter 23, which by itself is inspired by the proof of the divergence of the harmonic series:

\[
\sum_{n=2}^{2^K} \frac{1}{n \log n} = \sum_{k=1}^{K} \left( \sum_{n \colon 2^{k-1} < n \leq 2^k} \frac{1}{n \log n} \right) = \#.
\]

If we replace each of the inner sums here by the number of terms in it times the smallest of those, which is the last of those, it only becomes smaller. Hence

\[
\# > \sum_{k=1}^{K} 2^{k-1} \frac{1}{2^k \log 2^k} = \sum_{k=1}^{K} \frac{2^{k-1}}{2^k \log 2} = \frac{2}{\log 2} \sum_{k=1}^{K} \frac{1}{k}.
\]

The latter are partial sums of a divergent positive series, hence they approach infinity. Therefore the partial sums \( \sum_{n=2}^{2^K} \frac{1}{n \log n} \) approach infinity and our series is divergent.

**The results.** 75 students took the exam; the average grade is 47.4, the median is 46 and the standard deviation is 23.55.