In[1] := Trefoil = X[a, d, b, e] X[e, b, f, c] X[c, f, d, a]

Out[1] := X[a, d, b, e] X[c, f, d, a] X[e, b, f, c]

The definition of the Kauffman Bracket is given by the skein relations

\[
\langle \begin{array}{c}
\text{b} \\
\text{c} \\
\text{d} \\
\text{a}
\end{array} \rangle \rightarrow A \langle \begin{array}{c}
\text{a} \\
\text{c}
\end{array} \rangle + B \langle \begin{array}{c}
\text{b} \\
\text{d}
\end{array} \rangle
\]

and

\[
\langle \begin{array}{c}
\text{P} \\
\text{c}
\end{array} \rangle \rightarrow \text{d} \langle \text{P} \rangle
\]

The first rule can be coded as follows:

\[
\text{In[2]} := \text{rule1} = (X[a_, b_, c_, d_] :> A \delta[a d] \delta[b c] + B \delta[a b] \delta[c d]);
\]

Let us apply the first rule to the trefoil knot, just as an example:

\[
\text{In[3]} := \text{Trefoil} /. \text{rule1}
\]

Out[3] := (A \delta[b d] \delta[a e] + B \delta[a d] \delta[b e])  
     (A \delta[c e] \delta[b f] + B \delta[b e] \delta[c f])  
     (B \delta[a d] \delta[c f] + A \delta[a c] \delta[d f])

\[
\text{In[4]} := \text{Trefoil} /. \text{rule1} // \text{Expand}
\]

Out[4] := A^2 B \delta[a d] \delta[b d] \delta[a e] \delta[c e] \delta[b f] \delta[c f] + A B^2 \delta[a d]^2 \delta[b e] \delta[c e] \delta[b f] \delta[c f] +  
     A B^2 \delta[a d] \delta[b d] \delta[a e] \delta[b e] \delta[c f]^2 + B^3 \delta[a d]^2 \delta[b e]^2 \delta[c f]^2 +  
     A^2 B \delta[a c] \delta[b d] \delta[a e] \delta[b e] \delta[c e] \delta[b f] \delta[c f] + A B^2 \delta[a c] \delta[a d] \delta[b e] \delta[c e] \delta[b f] \delta[d f] +  
     A^2 B \delta[a c] \delta[b d] \delta[a e] \delta[b e] \delta[c f] \delta[d f] + A B^2 \delta[a c] \delta[a d] \delta[b e]^2 \delta[c f] \delta[d f]
\]

We see that we need a rule that will dispose of unnecessary intermediate points:

\[
\text{In[5]} := \text{rule2} = (\delta[a\_b\_] \delta[b\_c\_] :> \delta[a c]);
\]
In[6]:= (Trefoil /. rule1 // Expand) /. rule2


This done, we can code the second rule, (using dd instead of d, to avoid naming conflicts)

In[7]:= rule3 = ((δ[auty])^2 -> dd, δ[auty]^2 -> dd);

and see what happens to the trefoil knot:

In[8]:= RawBracket[t_] := Simplify[(t /. rule1 // Expand) /. rule2 /. rule3]

In[9]:= RawBracket[Trefoil]


Let’s verify the second Reidemeister move:

\[\begin{align*}
\text{In[10]} := \text{RawBracket}[X[b, e, f, a] X[f, e, c, d]] &= δ[a d] δ[b c] \\
\text{Out[10]} &= A B δ[b c] δ[a d] + (A^2 + B^2 + A B dd) δ[a b] δ[c d] \rightarrow δ[b c] δ[a d]
\end{align*}\]

Thus we see that for the second Reidemeister move to hold, we must have \( A B = 1 \) and \( A^2 + B^2 + A B d = 0 \). Solving for \( B \) and \( dd \) in terms of \( A \) we get:

\[\begin{align*}
\text{In[11]} := \text{rule4} &= \{ B \to 1/A, \ dd \to -A^2 - 1/A^2 \} \\
\text{Out[11]} &= \{ B \to 1/A, \ dd \to -A^2 - 1/A^2 \}
\end{align*}\]

Time to check the third Reidemeister move: (It’s guaranteed to hold, but lose nothing by checking again)

\[\begin{align*}
\text{In[12]} := \text{Bracket[t]} := \text{Simplify[RawBracket[t]/dd /. rule4]}
\end{align*}\]
The remaining sad point is the behavior of the bracket under the first Reidemeister move:

$$\begin{align*}
\langle \raisebox{-1.5em}{\includegraphics{bracket.png}} \rangle = \langle \begin{array}{c}
\text{a} \\
\text{c} \\
\text{b}
\end{array} \rangle
\end{align*}$$

In[15]:= Simplify[RawBracket[X[c, a, b, c]] /. rule4]

Out[15]= \(-A^3 \delta[a b]\)

This is unfortunate, but not fatal. It can be fixed by multiplying the bracket by \(-A^3\) raised to an appropriate multiple of the writhe is common to make the further substitution \(A \rightarrow q^{1/4}\) and then the resulting invariant is called the Jones polynomial. Let us

In[16]:= Jones[t_., w_]: = Simplify[(Bracket[t] \(-A^3\) \(-w\)) /. A \rightarrow q^(1/4)]

In[17]:= Jones[Trefoil, 3]

Out[17]= \(-1 + q + q^3\)

In[18]:= MirrorTrefoil = Trefoil /. x_ \rightarrow RotateLeft[x]

Out[18]= X[b, f, c, e] X[d, b, e, a] X[f, d, a, c]

In[19]:= Jones[MirrorTrefoil, -3]

Out[19]= \(q + q^3 - q^4\)

Ok, if you insist.

In[20]:= w[K_]: = Module[
{p, l},
  l = ReplacePart[K /. X[a_, b_, c_, d_] \rightarrow \delta[a c] \delta[b d], p, \{1, 0\}] //.
  {p[a b_] \rightarrow p[a, b], p[a_, b___, c_] \delta[c d_] \rightarrow p[a, b, c, d]};
  l = MapThread[Rule, (List @@ l, Range[Length[l]])];
  (Plus @@ (K /. l)) /.
  X[a_, b_, c_, d_] \rightarrow If[Abs[p = (a - c) (b - d)] = 1, p, -Sign[p]]
]

In[21]:= Jones[K_]: = Jones[K, w[K]]
In[22]:= \{Jones[Trefoil], Jones[MirrorTrefoil]\}

Out[22]= \\left\{ \frac{-1 + q + q^3}{q^4}, \ q + q^3 - q^4 \right\}

Now, last but not least.

In[23]:= Expand[\( \\
q \ast (-A^3) \ast (-1) \ast \text{RawBracket}[X[a, b, c, d]] \ast \\
q \ast (-1) \ast (-A^3) \ast \text{RawBracket}[X[b, c, d, a]] \)
\)/. \{A \rightarrow q^{(1/4)}, \ B \rightarrow q^{(-1/4)}\}

Out[23]= \frac{\delta[a \ b]}{\sqrt{q}} - \sqrt{q} \ \delta[a \ c] \ \delta[a \ d]