University of Toronto
Faculty of Arts and Science
MAT237Y1Y - Advanced Calculus
Term Test 2, Fall 2015
Duration - 1 hour and 50 minutes
No Aids Permitted

Last name: 

First Name: 

Student Number: 

Attention: → Make sure to attend the same tutorial that you check below when you pick up your test (in the 1st or 2nd week), in order to retain the “tutorial bump” bonus of +1 mark.

<table>
<thead>
<tr>
<th>Tutorial:</th>
</tr>
</thead>
<tbody>
<tr>
<td>T0201</td>
</tr>
<tr>
<td>M4</td>
</tr>
<tr>
<td>RS208</td>
</tr>
<tr>
<td>Travis</td>
</tr>
</tbody>
</table>

This exam contains 9 pages (including this cover page) and 7 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.

- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

- If you need more space, use the back of the pages; clearly indicate when you have done this.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
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1. (10 points) Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be defined as follows \( f(x, y, z) = \ln |ax + by^2 + cz^3| \), where \( a, b, c \) are real numbers. Find the gradient of \( f \) at the point \( P = (0, 1, -1) \).

Let \( \nabla f = \left[ \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right] \) be the gradient.

\[
\frac{\partial f}{\partial x} = (ax + by^2 + cz^3)^{-1} \cdot (a)
\]
\[
\frac{\partial f}{\partial y} = (ax + by^2 + cz^3)^{-1} \cdot (2by)
\]
\[
\frac{\partial f}{\partial z} = (ax + by^2 + cz^3)^{-1} \cdot (3cz^2)
\]

Thus, \( \nabla f = (ax + by^2 + cz^3)^{-1} \left[ a, 2by, 3cz^2 \right] \).

At \( P = (0, 1, -1) \):

We get \( \nabla f = (b - c)^{-1} \left[ a, 2b, 3c \right] \).
2. (10 points) Let $A$ and $B$ be path connected sets such that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is path connected.

*Solution:*

Suppose that $x$ and $y$ are in $A \cup B$. We will show that there is a path between them. If $x$ and $y$ are both in $A$ or both in $B$, then there is a path between them since $A$ and $B$ are path connected. Assume then, without loss of generality, that $x \in A$ and $y \in B$.

Since $A \cap B$ is nonempty, there exists $z \in A \cap B$. Since $x, z \in A$ and $A$ is path connected, there is a path $\gamma_1 : [0, 1] \to A$ such that $\gamma_1(0) = x$ and $\gamma_1(1) = z$. Similarly, since $z, y \in B$ and $B$ is path connected, there is a path $\gamma_2 : [0, 1] \to B$ such that $\gamma_2(0) = z$ and $\gamma_2(1) = y$.

Define a map (a concatenation of paths $\gamma_1$ and $\gamma_2$) $\gamma : [0, 1] \to A \cup B$ by

$$
\gamma(t) = \begin{cases} 
\gamma_1(2t) & 0 \leq t \leq 1/2 \\
\gamma_2(2t - 1) & 1/2 < t \leq 1.
\end{cases}
$$

Then $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma$ is continuous because

$$
\lim_{t_0 \to t_0} \gamma(t) = \begin{cases} 
\gamma_1(2t_0) = \gamma(t_0) & 0 \leq t_0 < 1/2 \\
\gamma_2(2t_0 - 1) = \gamma(t_0) & 1/2 < t_0 \leq 1 \\
\lim_{t \to 1/2^-} \gamma_1(2t) = \gamma(1/2) = z = \lim_{t \to 1/2^+} \gamma_2(2t - 1) & t_0 = 1/2.
\end{cases}
$$

Hence $\gamma$ is a path in $A \cup B$ between $x$ and $y$. 
3. (10 points) Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be defined as follows:

\[
f(x, y) = \begin{cases} 
  \frac{y}{x} & \text{if } x \neq 0 \\
  0 & \text{if } x = 0 
\end{cases}
\]

Find the partial derivatives \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) at the point \( P = (0, 0) \) if they exist or show they don't exist.

Hint: Use the definition of the partial derivative.

\[
\frac{\partial f}{\partial x}(p) = \lim_{t \to 0} \frac{f(x+t, y) - f(x, y)}{t} = \lim_{t \to 0} \frac{\frac{y}{x+t} - \frac{y}{x}}{t} = 0
\]

\[
\frac{\partial f}{\partial y}(p) = \lim_{t \to 0} \frac{f(x, y+t) - f(x, y)}{t} = \lim_{t \to 0} \frac{\frac{y+t}{x} - \frac{y}{x}}{t} = 0
\]
4. (10 points) Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be a function that is differentiable everywhere. Let \( g : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by \( g(x, y, z) = (u, v, w) = (x - y, y - z, z - x) \). Let \( h = f \circ g \). Show that \( \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} + \frac{\partial h}{\partial z} = 0 \).

Let \( M, A, B \) denote the Jacobian matrices of \( h, f, g \) respectively, (at appropriate pts).

\[
M = A \cdot B
\]

\[
B = \begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
-1 & 0 & 1
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w}
\end{bmatrix}
= \begin{bmatrix}
f_u & f_v & f_w
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
f_u - f_w & -f_u + f_v & -f_v + f_w
\end{bmatrix} = \begin{bmatrix} h_x & h_y & h_z \end{bmatrix}
\]

Notice the entries of \( M \) add to \( 0 \)
5. (10 points) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = \sqrt{|xy|} \). Prove that \( f \) is not differentiable at \( (0,0) \).

Assume \( f \) is differentiable at \( a = (0,0) \) and seek a contradiction.

Thus, by definition, \( \exists \) a matrix \( B = [b_1, b_2] \) such that:

\[
\lim_{h \to 0} \frac{|f(a+h) - f(a) - Bh|}{\|h\|} = 0. \quad (\text{Eqn } #1)
\]

In our case,

\[
\lim_{h \to 0} \frac{|f(h) - B'h|}{\|h\|} = 0 \quad (\text{Eqn } #2).
\]

1. Find \( B \) by either approaching \( h \to 0 \) along the two paths \( \chi_1 : h = t(1,0), \quad t \to 0 \)
   and \( \chi_2 : h = t(0,1), \quad t \to 0 \), respectively
   giving \( b_1 = 0; \quad b_2 = 0 \) by using eqn \#2.

   or

2. Find \( B \) by computing the partial derivative, from its definition. Also gives \( B = [0 \ 0] \)

   Thus, eqn \#2 becomes:

\[
\lim_{h \to 0} \frac{|f(h)|}{\|h\|} = 0. \quad \text{But approaching along path } \chi_3 : h = t(1,1), \quad t \to 0 \quad \text{will result in } \frac{1}{\sqrt{2}} = 0 \quad \text{in } A. \quad \text{contradiction.}.
\]
6. Let $f : [-2, 2] \rightarrow \mathbb{R}$ be the continuous function defined as $f(x) = x^2$.

   (a) (2 points) Show that $f$ is uniformly continuous by stating the appropriate theorem.

   (b) (8 points) Prove using the $(\epsilon, \delta)$ definition that $f$ is uniformly continuous.

Solution:

(a) Any continuous map with a compact domain is uniformly continuous. Since $f$ is continuous and $[-2, 2]$ is compact (closed and bounded), then $f$ is uniformly continuous.

(b) Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{4}$. Note that $\delta$ only depends on $\varepsilon$. If $x, y \in [-2, 2]$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y| < \delta|x + y| = \frac{\varepsilon}{4}|x + y| \leq \varepsilon$$

since $|x + y| \leq |x| + |y| \leq 4$. 
7. (10 points) Suppose that for each $n \in \mathbb{N}$, $X_n \neq \emptyset$ is a compact set and $X_{n+1} \subset X_n$. Prove that $igcap_{n \in \mathbb{N}} X_n \neq \emptyset$. Hint: Let $x_n \in X_n$ and use the Bolzano-Weierstrass Theorem.

**Solution:**

Since $X_n$ is nonempty for each $n \in \mathbb{N}$, there exists a point $x_n \in X_n$. Since $X_{n+1} \subset X_n$ for all $n \in \mathbb{N}$, $x_n \in X_1$ for all $n \in \mathbb{N}$. Since $X_1$ is compact, by the Bolzano-Weierstrass theorem, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$ which converges to a point $x$ in $X_1$.

**Claim:** For any $m \in \mathbb{N}$, $x \in X_m$.

**Proof:** The sequence $(x_{n_{m+k}})_k = x_{n_{m+1}}, x_{n_{m+2}}, x_{n_{m+3}}, \ldots$ is contained inside $X_{n_{m+1}}$ and hence in $X_m$ since $m < n_{m+1}$. This sequence, being a subsequence of $(x_{n_k})_k$ converges to $x$ as well. Any convergent sequence in $X_m$ converges to a point in $X_m$ since $X_m$ is closed. Hence $x \in X_m$.

We conclude that $x \in \bigcap_{n \in \mathbb{N}} X_n$, and so $\bigcap_{n \in \mathbb{N}} X_n$ must be nonempty.