15.0 Green’s Theorem and Conservative Vector Fields

⋆1. (a) Determine the values of $\alpha$ and $\beta$ such that

$$\mathbf{F}(x, y, z) = (y + z, \alpha x + z, x + \beta y)$$

is a conservative vector field. Write down the potential function $f : \mathbb{R}^3 \to \mathbb{R}$ so that $\mathbf{F} = \nabla f$.

(b) Let $C$ be the curve given parametrically by

$$\gamma(t) = (t \cos(t), t \sin(t), t^2), \quad 0 \leq t \leq \frac{\pi}{2}$$

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ where $\mathbf{F}$ is your solution from part (a).

⋆2. Compute the line integral of curve $\{x^2 + y^2 = 1, y \geq 0\}$ (oriented counter clockwise) through the vector field $\mathbf{F}(x, y) = (2xy^2 + 1, 2x^2y + 2y)$.

⋆3. (a) Show that the vector field $\mathbf{F}(x, y, z) = (y \cos(xy), x \cos(xy) + e^z, ye^z)$ is conservative and find its scalar potential.

(b) Consider the curve $C$ given by the intersection of the sets

$$C = \{x^2 + y^2 + z^2 = 4\} \cap \{x = 0\} \cap \{z \geq 0\}$$

oriented so that its tangent point is the positive $y$-direction. Determine $\int_C \mathbf{F} \cdot d\mathbf{x}$ if $\mathbf{F}$ is the vector field given in part (a).

⋆4. Evaluate the following line integrals both directly and by using Green’s Theorem.

(a) $\iint_C (x + 2y)dx + (x - 2y)dy$ where $C$ is given by the union of the images of the following two functions on $[0, 1]$: $f(x) = x^2$ and $g(x) = x$, positively oriented with respect to the area the curves bound.

(b) $\iint_C (3x - 5y)dx + (x + 6y)dy$ where $C$ is the ellipse $x^2/4 + y^2 = 1$ oriented counter-clockwise.

⋆⋆5. (a) Let $D \subseteq \mathbb{R}^2$ be a regular region and $\partial D = C$ be a piece-wise smooth simple closed curve, oriented positively. If $A(D)$ is the area of $D$, show that

$$A(D) = \iint_C xy = \int_C ydx = \int_C \frac{1}{2} (ydx - xdy).$$

(b) Consider the disk of radius $r$, $D_r = \{x^2 + y^2 \leq r\} \subseteq \mathbb{R}^2$. Use any of the formulae from part (a) to compute the area of this disk. [Of course, you already know what the result should be!]

(c) In this question we will show that artificially adding boundaries does not affect the line integral. Let $L_r$ be any diameter of $D$. Show that if we break $D$ into two regions $D = D_1 \cup D_2$ and give the boundary of $D_1$ and $D_2$ positive orientations, then for any $C^1$ vector field $\mathbf{F}(x, y)$ we have

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{x} = \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{x} + \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{x}.$$
(d) Use part (a) to compute the area of the lemniscate \( x^4 = x^2 - y^2 \). [Be careful, as the lemniscate’s boundary is not a simple closed curve.]

**6.** For this problem, we will always be working in \( \mathbb{R}^3 \). Let \( \Omega^0(\mathbb{R}^3) \) be the set of \( C^1 \) functions \( \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( \Omega^1(\mathbb{R}^3) \) be the set of \( C^1 \) vector fields \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \).

(a) Show that \( \Omega^0(\mathbb{R}^3) \) and \( \Omega^1(\mathbb{R}^3) \) are both \( \mathbb{R} \)-vector spaces. [This should be short proof!]

(b) Show that \( \text{grad} : \Omega^0 \rightarrow \Omega^1 \) and \( \text{curl} : \Omega^1 \rightarrow \Omega^1 \) are linear operators.

(c) Recall that if \( Z \) is the set of closed vector fields, then \( Z = \ker(\text{curl}) \) and if \( B \) is the set of exact vector fields, then \( B = \text{im}(\text{grad}) \). Show that \( B \subseteq Z \).