1 Differentiability

⋆1. Compute the partial derivatives \( \partial_i f, \) and the derivative \( Df \) (in matrix form) for each of the following functions \( f : \mathbb{R}^n \to \mathbb{R}^k \):

(a) \( f(x, y, z) = x^y \)
(b) \( f(x, y) = x^y \)
(c) \( f(x, y, z) = (x^y, z) \)
(d) \( f(x, y) = \sin(x \sin(y)) \)
(e) \( f(x, y, z) = (x + y)^2 \)
(f) \( f(x, y, z) = (\log(x^2 + y^2 + z^2), xyz) \)
(g) \( f(x, y) = \sin(xy) \)

⋆2. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map \( f(x, y) = (x^2 - y^2, 2xy) \).

(a) Calculate \( Df \) and \( \det Df \)

⋆3. Suppose that \( f : \mathbb{R}^3 \to \mathbb{R}^2 \) is a function such that \( f(0, 0, 0) = (1, 2) \) and:

\[
Df_{(0,0,0)} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}
\]

Let \( g : \mathbb{R}^2 \to \mathbb{R}^2 \) be the map \( g(x, y) = (x + 2y + 1, 3xy) \). Find \( D(f \circ g)_{(0,0,0)} \)

⋆4. Find an equation for the tangent plane \( T_pS \) to the following surfaces at the indicated point:

(a) \( S = \{(x, y, z) \mid x^2 + 2y^2 + 3z^2 = 6\} \) at \((1,1,-1)\).
(b) \( S = \{(x, y, z) \mid xyz^2 - \log(z - 1) = 8\} \) at \((-2,-1,2)\).
(c) \( S = \{(x, y, z) \mid x^2 + y^2 = 1\} \) at \((1/\sqrt{2}, 1/\sqrt{2}, 1)\)

⋆5. Suppose that \( f(x, y, z, t), x(t), y(x, t, s), \) and \( z(y, x) \). Use the chain rule to find an expression for \( \frac{\partial f}{\partial t} \) and \( \frac{\partial f}{\partial s} \).

⋆⋆6. Show that \( f : \mathbb{R}^2 \to \mathbb{R}, f(x, y) = \sqrt{|xy|} \) is not differentiable at \((x, y) = (0,0)\).

⋆⋆7. Let \( f, g : \mathbb{R}^n \to \mathbb{R}^m \). If \( f \) and \( g \) are differentiable at \( x \in \mathbb{R}^n \), then \( D(f + g) = Df_x + Dg_x \) and \( D(fg)_x = f(x)Dg_x + g(x)Df_x \). Notice these are generalizations of the sum and product rules for differentiation from last year. (Hint: Notice that the maps \((x, y) \to x + y \) and \((x, y) \to xy \) are themselves differentiable maps)

⋆⋆8. If \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable and \( \nabla f(x) = 0 \) for all \( x \), then \( f \) is constant.