1.3 Continuity

*1. Find the limit, if it exists, or prove that the limit does not exist

(a) \( \lim_{(x,y) \to (0,0)} (5x^3 - x^2 y^2) \)
(b) \( \lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \) This one is a bit harder. Hint: Bound the Euclidian norm using the sup norm.
(c) \( \lim_{(x,y) \to (0,0)} \frac{x^4 - 4y^2}{x^2 + 2y^2} \)
(d) \( \lim_{(x,y) \to (0,0)} \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} \)
(e) \( \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2(y)}{x^2 + 2y^2} \). This one is also hard. Try and bound \( \sin(y) \) by a function which is easier to understand.

**2. Define a function \( f : \mathbb{R}^2 \setminus \{(x, y) \mid x = 0\} \to \mathbb{R} \) as follows:

\[
 f(x, y) = \frac{\sin(xy)}{x}
\]

How should you define \( f(x, y) \) at \( x = 0 \) so that \( f(x, y) \) extends to a continuous function on all of \( \mathbb{R}^2 \)?

**3. Prove the following are equivalent:

(a) \( f : \mathbb{R}^m \to \mathbb{R}^n \) is \( \epsilon, \delta \) continuous
(b) For every \( V \subseteq \mathbb{R}^n \) open, \( f^{-1}(V) \) is open
(c) For every \( V \subseteq \mathbb{R}^n \) open, \( \forall x \in \mathbb{R}^m \), if \( f(x) \in V \) then there exists an open set \( U \subseteq \mathbb{R}^m \) containing \( x \) such that \( f(U) \subseteq V \).

*4. Find an example of a continuous function \( f : \mathbb{R}^n \to \mathbb{R}^m \) and an open set \( U \subseteq \mathbb{R}^n \) such that \( f(U) \) is not open. Suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \) and there exists a continuous \( g : \mathbb{R}^n \to \mathbb{R}^m \) such that \( f(g(x)) = x \) and \( g(f(x)) = x \). If \( U \) is open, must \( f(U) \) be open? Must \( g(U) \) be open?

**5. If \( f : S \to \mathbb{R} \) is uniformly continuous and \( \{x_k\} \) is a Cauchy sequence, then \( \{f(x_k)\} \) is Cauchy

**6. If \( f : S \to \mathbb{R} \) is uniformly continuous, then there exists a unique continuous function \( \tilde{f} : \overline{S} \to \mathbb{R} \) such that \( \tilde{f}|_S = f \)

**7. Let \( f, g : \mathbb{R}^n \to \mathbb{R}^k \) be continuous functions and suppose that \( D \subseteq \mathbb{R}^n \) is a dense set. If \( f(x) = g(x) \) for every \( x \in D \), then \( f(x) = g(x) \) for every \( x \in \mathbb{R}^n \).

**8. Let \( C([0, 1]) = \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous on } [0, 1] \} \) be the set of all continuous functions on the closed unit interval. Define the following function:

\[
 d : C([0, 1]) \times C([0, 1]) \to \mathbb{R}
\]

\[
 d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|
\]

Show that \( d \) has the following properties:

(a) \( d(f, g) \geq 0 \), and \( d(f, g) = 0 \) if and only if \( f(x) = g(x) \) for every \( x \in [0, 1] \)
(b) \( d(f, h) \leq d(f, g) + d(g, h) \) \( \forall f, g, h \in C([0, 1]) \)
(c) \( d(f, g) = d(g, f) \) \( \forall f, g \in C([0, 1]) \)

We say that \( d \) defines a metric on \( C([0, 1]) \). We can use \( d \) to make sense of open sets in \( C([0, 1]) \). Fix a continuous function \( g \in C([0, 1]) \), and let \( U = \{f \in C([0, 1]) \mid f(x) < g(x) \forall x \in [0, 1] \} \). Show that \( U \) is an open set with respect to the metric \( d \).

Is \( U \) still open if instead of \( C([0, 1]) \), we had started with \( C(\mathbb{R}) \), continuous functions on \( \mathbb{R} \), and set \( U = \{f \in C(\mathbb{R}) \mid f(x) < g(x) \forall x \in \mathbb{R} \} \)?