1 Topology of \( \mathbb{R}^n \)

1.1 Basic Set Theory

**1.** Let \( A \subseteq S \) and \( B \subseteq S \). Prove each of the following statements

(a) \( A \subseteq B \) if and only if \( A \cup B = B \)
(b) \( A^c \subseteq B \) if and only if \( A \cup B = S \)
(c) \( A \subseteq B \) if and only if \( B^c \subseteq A^c \)
(d) \( A \subseteq B^c \) if and only if \( A \cap B = \emptyset \)

**2.** Let \( A, B, \) and \( C \) be subsets of \( S \). Show that if \( A \subseteq B \) and \( B \subseteq C \), then \( A \subseteq C \).

**3.** Let \( A_1, A_2, \ldots, A_n \) be sets. If \( A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_n \) and \( A_n \subseteq A_1 \), then \( A_1 = A_2 = A_3 = \cdots = A_n \).

**4.** Let \( I \) be an index for a collection of subsets \( A_i \subseteq S, i \in I \). Show that for every \( k \in I \), \( \bigcap_{i \in I} A_i \subseteq A_k \).

**5.** Let \( f : A \to B \) be a function.

(a) For every \( X \subseteq A \), \( X \subseteq f^{-1}(f(X)) \)
(b) For every \( Y \subseteq B \), \( Y \supseteq f(f^{-1}(Y)) \)
(c) If \( f : A \to B \) is injective, then for every \( X \subseteq A \) we have \( X = f^{-1}(f(X)) \)
(d) If \( f : A \to B \) is surjective, then for every \( Y \subseteq B \) we have \( Y = f(f^{-1}(Y)) \)

**6.** Let \( f : A \to B \) be a map of sets, and let \( \{X_i\}_{i \in I} \) be an indexed collection of subsets of \( A \).

(a) Prove that \( f \left( \bigcup_{i \in I} X_i \right) = \bigcup_{i \in I} f(X_i) \)
(b) Prove that \( f \left( \bigcap_{i \in I} X_i \right) \subseteq \bigcap_{i \in I} f(X_i) \)
(c) When does equality of sets hold in the above part?

1.2 Open and Closed Sets

**1.** Let \( \mathbf{u}, \mathbf{v} \) be two vectors in \( \mathbb{R}^3 \). Compute \( \mathbf{u} \times \mathbf{v}, \mathbf{u} \cdot \mathbf{v} \) for the following choices of \( \mathbf{u} \) and \( \mathbf{v} \), and state whether or not \( \mathbf{u} \) and \( \mathbf{v} \) are either orthogonal or colinear.

(a) \( \mathbf{u} = (6, 0, -2), \mathbf{v} = (0, 8, 0) \)
(b) \( \mathbf{u} = (1, 1, -1), \mathbf{v} = (2, 4, 6) \)

**2.** Let \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3 \) be vectors. Determine which of the following expressions are meaningless:

(a) \( \langle \mathbf{a} \cdot \mathbf{b}, \mathbf{c} \rangle \)
(b) \( |\mathbf{a}| \langle \mathbf{b} \cdot \mathbf{c} \rangle \)
(c) \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) \)
(d) \( \langle \mathbf{a} \cdot \mathbf{b} \rangle \mathbf{c} \)
(e) \( \mathbf{a} \cdot \mathbf{b} + c \)
(f) \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \)
(g) \( \langle \mathbf{a} \cdot \mathbf{b} \rangle \cdot (\mathbf{c} \cdot \mathbf{d}) \)
(h) \( \langle \mathbf{a} \times \mathbf{b} \rangle \cdot (\mathbf{c} \times \mathbf{d}) \)

**3.** Prove that for any \( x, y \in \mathbb{R}^n \), \( \|x - y\| \geq \|x\| - \|y\| \). This is commonly called, “the reverse triangle inequality”. It is extremely useful when you want to prove inequalities like \( \|(x - a)^{-1}\| \leq M \) for \( x \) in some fixed closed set.
**4.** Can a set be both open and closed? Prove or disprove. Can a set be neither closed nor open? Prove or disprove. The point of this question is to get you to understand that “not closed ≠ open” (and conversely, “not open ≠ closed”). See also this humorous instructional video (Warning: Strong language)

**5.** If $U$ and $V$ are open (resp. closed) then $U \cup V$ is open (resp. $U \cap V$ is closed). If $\{U_i\}_{i=1}^{\infty}$ is a countable collection of open sets, must $\bigcap_{i \in I} U_i$ be open? Provide a proof or counterexample. Similarly, if $\{A_i\}_{i \in I}$ is an infinite collection of closed sets, must $\bigcap_{i \in I} A_i$ be closed?

**6.** Prove that the following sets are open

(a) $\mathbb{R}^n$
(b) $B(r, x)$
(c) $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$
(d) $\{(x, y) \in \mathbb{R}^2 \mid x > 1$ and $y > 0\}$ (Hint: You could do this by brute force, or notice that the set can be written as an intersection of two other sets)
(e) $\{(x, y) \in \mathbb{R}^2 \mid x \notin \mathbb{Z}\}$ (This one is a bit harder)

**7.** Determine and prove whether the following sets are open, closed, or neither open nor closed

(a) $\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Z}\}$
(b) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
(c) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$
(d) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, (x, y) \neq (0, 0)\}$
(e) $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y = \sin(1/x)\}$ (Hint: Plot this beast, then look at what happens as you approach $x = 0$ along lines $y = C$)
(f) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, 0 < y < 1, x, y \in \mathbb{Q}\}$ (Hint: You might need to wait until we study completeness to answer this)
(g) $\{(x, y) \in \mathbb{R}^2 \mid y > x^2\}$

**8.** If $S$ is not closed, then there exists $x \in \overline{S}$ but $x \notin S$

**9.** If $S \subseteq \mathbb{R}^n$ is not closed, then there exists a sequence $\{x_k\}_{k=1}^{\infty} \subseteq S$ such that no subsequence of $\{x_k\}$ converges in $S$.

**10.** Show that $B(r, x) \subseteq B(r + ||x||, 0)$ - i.e. the definition of a bounded set did not depend on where we centre our ball.

**11.** Construct an open set which contains the rational numbers $\mathbb{Q}$, but which is a proper subset of $\mathbb{R}$.

**12.** Prove that $\overline{S} = \bigcap_{A \supseteq S, A \text{ closed}} A$ - i.e. the closure of $S$ is the intersection of all the closed sets containing $S$.

**13.** The intent of this exercise is to show that if we were to start the course over and use open squares instead of open balls in defining open sets, we would have actually had the same definition! Define a map

$$|| \cdot \parallel_{\infty} : \mathbb{R}^n \rightarrow \mathbb{R}$$

$$||x||_{\infty} = \max_{i} \{|x_i|\}$$

Where $x = (x_1, \ldots, x_i, \ldots, x_n)$. For any $a \in \mathbb{R}^n$, let $S(a, \epsilon) = \{x \in \mathbb{R}^n \mid ||x - a||_{\infty} < \epsilon\}$. We say that a set $U$ is $S$-open (the $S$ stands for square) if and only if for every $a \in U$, there exists $\epsilon > 0$ such that $S(a, \epsilon) \subseteq U$.

(a) Make a sketch of $S(0, 1)$ when $n = 2$.

(b) Show that for any $x \in \mathbb{R}^n$, $||x||_{\infty} \leq ||x|| \leq \sqrt{n}||x||_{\infty}$, where $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$
(c) Prove that \( U \subseteq \mathbb{R}^n \) is S-open if and only if \( U \) is open.

(d) Consider the functions \( \| \cdot \|_p : \mathbb{R}^n \to \mathbb{R}, \|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p} \), where \( 1 < p < \infty \). Plot the sets (by hand, or using a computer) \( \{ x \in \mathbb{R}^2 | \|x\|_p < 1 \} \). Do you expect the \( p \)-balls to define the same collection of open sets as the 2-balls? Explain (proof is not necessary).