Problem 1

\[(t^2 + 1) xu_t + (x^2 + 1) tu_x = 0.\]  
(1)

(a) Find the characteristic curves and sketch them in the \((x, t)\) plane.

(b) Write the general solution.

(c) Where the solution is fully determined by the initial condition \(u(x, 0) = g(x)\).

(d) Solve equation (1) with the initial condition \(u(x, 0) = x^2\).

(e) Solve equation (1) with the initial condition \(u(x, 0) = x\).

Problem 2 Solve the wave equation with the following initial conditions

\[
\begin{cases}
  u_{tt} - 25u_{xx} = e^{-x-t}, & -\infty < x < \infty \\
  u(x, 0) = xe^{-x}, \\
  u_t(x, 0) = e^{-x}.
\end{cases}
\]  
(2)

Problem 3 Consider wave equation with boundary conditions:

\[
\begin{cases}
  u_{tt} - c^2 u_{xx} + \gamma u = 0, & 0 < x < L, \\
  (u_x - \alpha u_t)(0, t) = 0, \\
  (u_x - \beta u_t)(L, t) = 0,
\end{cases}
\]  
(3)

with \(\alpha, \beta \in \mathbb{C}, \gamma \in \mathbb{R}\) and \(c > 0\).

Therefore \(u\) is a complex-valued function. Consider an energy \(E(t)\) defined as

\[
E(t) = \frac{1}{2} \int_0^L \left(|u_t|^2 + c^2 |u_x|^2 + \gamma |u|^2\right) dx.
\]  
(4)

(a) Find conditions to \(\alpha, \beta, \gamma\) such that \(E(t)\) does not depend on \(t\) for any \(u\) satisfying (3):
(b) Find conditions to $\alpha, \beta, \gamma$ such that $E(t)$ is non-increasing function of $t$ for any $u$ satisfying (3);

**Hint.** Each end is independent and conditions are separate for $\alpha$, and for $\beta$.

**Problem 4** Find solutions $u$ of IVP for a heat equation

$$\begin{cases}
    u_t - u_{xx} = 0 & -\infty < x < \infty, \\
    u(0, x) = f(x)
\end{cases}$$

where

(a) $f(x) = \theta(x) := \begin{cases}
    0 & x < 0, \\
    1 & x > 0;
\end{cases}$

(b) $f(x) = \begin{cases}
    0 & |x| < 1, \\
    1 & |x| > 1.
\end{cases}$

**Hint.** In (b) represent $f(x)$ as $\theta(x + 1) - \theta(x - 1)$.

**Remark.** Solution must be expressed through

$$\text{erf}(z) = \sqrt{\frac{2}{\pi}} \int_0^z e^{-s^2} ds$$