Problem 1  Consider equation with the initial conditions
\[ u_{tt} - 4u_{xx} = 0, \quad t > 0, x > vt, \]
\[ u|_{t=0} = e^{-x}, \quad x > 0, \]
\[ u_t|_{t=0} = e^{-x}, \quad x > 0, \]
(1)
(2)
(3)

(A) Let \( v = 3 \). Find which of these conditions (a)-(c) at \( x = vt, t \geq 0 \) could be added to (1)-(3) so that the resulting problem would have a unique solution:

(a) None,
(b) \( u|_{x=vt} = 0 \ (t \geq 0) \),
(c) \( u|_{x=vt} = u_x|_{x=vt} = 0 \ (t \geq 0) \).

Solve the problem you deemed as a good one.

(B) Let \( v = 1 \). Find which of these conditions (a)-(c) at \( x = vt, t \geq 0 \) could be added to (1)-(3) so that the resulting problem would have a unique solution:

(a) None
(b) \( u|_{x=vt} = 0 \ (t \geq 0) \),
(c) \( u|_{x=vt} = u_x|_{x=vt} = 0 \ (t \geq 0) \).

Solve the problem you deemed as a good one.

(C) Let \( v = -3 \). Find which of these conditions (a)-(c) at \( x = vt, t \geq 0 \) could be added to (1)-(3) so that the resulting problem would have a unique solution:

(a) None
(b) \( u|_{x=vt} = 0 \ (t \geq 0) \),
(c) \( u|_{x=vt} = u_x|_{x=vt} = 0 \ (t \geq 0) \).

Solve the problem you deemed as a good one.
Problem 2  A spherical wave is a solution of the three-dimensional wave equation of the form \( u(r,t) \), where \( r \) is the distance to the origin (the spherical coordinate). The wave equation takes the form

\[ u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right) \]  ("spherical wave equation").  \hspace{1cm} (4)

(a) Change variables \( v = ru \) to get the equation for \( v \): \( v_{tt} = c^2 v_{rr} \).

(b) Solve for \( v \) using

\[ v = f(x + ct) + g(x - ct) \]  \hspace{1cm} (5)

and thereby solve the spherical wave equation.

(c) Use

\[ v(r,t) = \frac{1}{2} \left[ \phi(r + ct) + \phi(r - ct) \right] + \frac{1}{2c} \int_{r-ct}^{r+ct} \psi(s) \, ds \]  \hspace{1cm} (6)

with \( \phi(r) = v(r,0) \), \( \psi(r) = v_t(r,0) \) to solve it with the initial conditions \( u(r,0) = \Phi(r) \), \( u_t(r,0) = \Psi(r) \).

(d) Find the general form of solution continuous as \( r = 0 \).

Problem 3

By method of continuation combined with D’Alembert formula solve each of the following four problems (a)–(d).

\[
\begin{cases}
  u_{tt} - 4u_{xx} = 0, & x \geq 0, \\
  u|_{t=0} = 0, & x \geq 0, \\
  u_t|_{t=0} = 1, & x \geq 0, \\
  u|_{x=0} = 0, & t \geq 0.
\end{cases} \tag{7}
\]

\[
\begin{cases}
  u_{tt} - 4u_{xx} = 0, & x \geq 0, \\
  u|_{t=0} = 0, & x \geq 0, \\
  u_t|_{t=0} = 1, & x \geq 0, \\
  u_x|_{x=0} = 0, & t \geq 0.
\end{cases} \tag{8}
\]

\[
\begin{cases}
  u_{tt} - 4u_{xx} = 0, & x \geq 0, \\
  u|_{t=0} = 0, & x \geq 0, \\
  u_t|_{t=0} = x, & x \geq 0, \\
  u|_{x=0} = 0, & t \geq 0.
\end{cases} \tag{9}
\]
\begin{align*}
\begin{cases}
  u_{tt} - 4u_{xx} = 0, & \text{ } x \geq 0, \\
  u|_{t=0} = 0, & \text{ } x \geq 0, \\
  u_t|_{t=0} = x, & \text{ } x \geq 0, \\
  u_x|_{x=0} = 0, & \text{ } t \geq 0.
\end{cases}
\end{align*} \quad (10)

\textbf{Problem 4}

For a solution \( u(x, t) \) of the wave equation with \( \rho = T = 1 \Rightarrow c = 1 \), the \textit{energy density} is defined as 
\[ e = \frac{1}{2}(u_t^2 + u_x^2) \]
and the \textit{momentum density} as 
\[ p = u_t u_x. \]

1. Show that 
\[ \frac{\partial e}{\partial t} = \frac{\partial p}{\partial x} \quad \text{and} \quad \frac{\partial p}{\partial t} = \frac{\partial e}{\partial x}. \] \quad (11)

2. Show that both \( e(x, t) \) and \( p(x, t) \) also satisfy the wave equation.