Deadline Monday, September 24, 9 pm

APM 346 (2012) Home Assignment 1

Problem 1

(a) Find general solution

\[ u_x - 3u_y = 0; \quad (1) \]

(b) Solve IVP problem \( u|_{x=0} = e^{-y^2} \) for equation (1) in \( \mathbb{R}^2 \);

(c) Consider equation (1) in \( \{ x>0, y>0 \} \) with the initial condition \( u|_{x=0} = y \ (y>0) \); where this solution defined? Is it defined everywhere in \( \{ x>0, y>0 \} \) or do we need to impose condition at \( y = 0 \)? In the latter case impose condition \( u|_{y=0} = x \ (x>0) \) and solve this IVBP;

(d) Consider equation (1) in \( \{ x<0, y>0 \} \) with the initial condition \( u|_{x=0} = y \ (y>0) \); where this solution defined? Is it defined everywhere in \( \{ x<0, y>0 \} \) or do we need to impose condition at \( y = 0 \)? In the latter case impose condition \( u|_{y=0} = x \ (x<0) \) and solve this IVBP.

Problem 2

(a) Find the general solution of

\[ xu_x + 4yu_y = 0 \quad (2) \]

in \( \{ (x, y) \neq (0, 0) \} \); when this solution is continuous at \( (0, 0) \)?

(b) Find the general solution of

\[ xu_x - 4yu_y = 0 \quad (3) \]

in \( \{ (x, y) \neq (0, 0) \} \); when this solution is continuous at \( (0, 0) \)?

(c) Explain the difference.

Problem 3  Find the solution of

\[
\begin{cases}
  u_x + 3u_y = xy, \\
  u|_{x=0} = 0.
\end{cases} \quad (4)
\]
Problem 4
(a) Find the general solution of
\[ yu_x - xu_y = xy; \]  \hspace{1cm} (5)
(b) Find the general solution of
\[ yu_x - xu_y = x^2 + y^2; \]  \hspace{1cm} (6)
(c) In one instance solution does not exist. Explain why.

Problem 5
(a) Find the general solution of
\[ u_{tt} - 9u_{xx} = 0; \]  \hspace{1cm} (7)
(b) Solve IVP
\[ u|_{t=0} = x^2, \quad u_t|_{t=0} = x \]  \hspace{1cm} (8)
for (7);
(c) Consider (7) in \( \{x > 3t, x > -3t\} \) and find a solution to it, satisfying
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\[ u|_{x=3t} = t, \quad u|_{x=-3t} = 2t. \]  \hspace{1cm} (9)

Problem 6 Derivation of a PDE describing traffic flow. The purpose of this problem is to derive a model PDE that describes a congested one-dimensional highway. Let
\( \rho(x, t) \) denote the traffic density: the number of cars per kilometer at time \( t \) located at position \( x \);
\( q(x, t) \) denote the traffic flow: the number of cars per hour passing a fixed place \( x \) at time \( t \);
\( N(t, a, b) \) denote the number of cars between position \( x = a \) and \( x = b \) at time \( t \).

Derive a formula for \( N(t, a, b) \) as an integral of the traffic density. You can assume there are no exits or entrances between position \( a \) and \( b \).

Derive a formula for \( \frac{\partial N}{\partial t} \) depending on the traffic flow.
HINT: You can express the change in cars between time $t_1 = t$ and $t_2 = t + h$ in terms of traffic flow;

Differentiate with respect to $t$ the integral form for $N$ from part (a) and make it equal to the formula you got in part (b). This of the integral form of conservation of cars;

Express the right hand side of the formula of part (c) in terms of an integral. Since $a, b$ are arbitrary, obtain a PDE. This PDE is called the conservation of cars equation;

What equation do you get in part (4) if $q = c\rho$, for some constant $c$. What choice of $c$ would be more realistic, i.e. what should $c$ be function of?