Non-Linear DFE

Exercise 1 \( Y_n Y_{n+2} = Y^2 \quad Y_0 = 1, \quad Y_1 = 2 \)

Follows that \( Y_n \neq 0 \ \forall n \).

\[ \therefore \quad \frac{Y_{n+2}}{Y_{n+1}} = \frac{Y_{n+1}}{Y_n} \]

Let \( W_n = \frac{Y_{n+1}}{Y_n} \). Then \( W_{n+1} = W_n, \quad W_0 = 2 \)

\[ \therefore W_n = 2 \rightarrow Y_{n+1} = 2Y_n = Y_n = 2^n \quad (Y_0 = 1) \]

N.B. Could also linearize with logs.

Exercise 2 \( Y_{n+2} = 2Y_n + 3Y_{n+1} - 4Y_n = 0 \)

Set \( W_n = Y_n^2 \)

\[ W_{n+2} + 3W_{n+1} - 4W_n = 0 \]

\[ W_n = c_1(-4)^n + c_2 \rightarrow Y_n = \sqrt{c_1(-4)^n + c_2} \]

Solving Recurrences Using Generating Fn

*Basic Idea: G.F. is formal power series with the coeff of interest to us related by the recursion. Using formal manipulations this leads to a formal expression (hopefully a recognizable closed form) from which the coefficients of the FPS can be determined.

Exercise: \( a_{n+1} = 2a_n + 1 \quad a_0 = 0, \quad a_1 = 1 \)

Define: \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) ("FPS")

\[ A(x) = a_0 + \sum_{n=1}^{\infty} a_n x^n \]

"Formal manipulation"

\[ = a_0 + \sum_{n=0}^{\infty} a_{n+1} x^{n+1} \]

\[ = \sum_{n=0}^{\infty} (2a_n + 1)x^{n+1} \]

\[ = 2 \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} x^{n+1} \]
\[ A(x) = 2x + \sum_{n \geq 0} a_n x^n = 2x A(x) + \frac{x}{1 - x} = \frac{x}{1 - x} + A(x) \]

\[ A(x) = \frac{x}{(1 - x)(1 - 2x)} \]

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\[ \begin{align*}
\sum_{n \geq 0} a_n x^n &= \sum_{n \geq 0} \left( \sum_{j=0}^{n} 2^j \right) x^n \\
\sum_{n \geq 0} a_n x^n &= \sum_{n \geq 0} (2^{n+1} - 1) x^{n+1}
\end{align*} \]
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Comparing coefficients we see that \( a_n = 2^n - 1 \)

**N.B.** If 2 FPS are equal, they are the same coefficient by coefficient.

**Exercise:** \( F_{n+2} = F_{n+1} + F_n \quad n \geq 0 \), \( F_0 = F_1 = 1 \)

Let \( F(x) = \sum_{n \geq 0} F_n x^n \) “FPS”

\[ F(x) = F_0 + F_1 x + \sum_{n \geq 2} F_n x^n \]

\[ = 1 + x + \sum_{n \geq 0} F_{n+2} x^{n+2} \]

\[ = 1 + x + \sum_{n \geq 0} F_{n+1} x^{n+1} + x^2 \sum_{n \geq 0} F_n x^n \]

\[ = 1 + x + \sum_{n \geq 0} (F_{n+1} + F_n) x^{n+2} \]

\[ = 1 + x + x [F(x) - F_0] + x^2 F(x) \]

\[ \therefore F(x) = 1 + x F(x) + x^2 F(x) \]

\[ \therefore F(x) = \frac{1}{1 - x - x^2} \]

How can we determine the coeff of the FPS for \( F(x) \) from this?

Use Partial Fraction Expansion of RHS!
Sometimes we use e.g.f. just as successfully.

\[ F(x) = \sum_n \frac{F_n}{n!} x^n \]

\[ = 1 + x + \sum_{n \geq 0} \frac{F_{n+2}}{(n+2)!} x^{n+2} \]

\[ = 1 + x + \sum_{n \geq 0} \frac{F_{n+1} + F_n}{(n+2)!} x^{n+2} \]

\[ \therefore F'(x) = 1 + \sum_{n \geq 0} \frac{F_{n+1} + F_n}{(n+1)!} x^{n+1} \]

\[ \therefore F''(x) = F'(x) + \sum_{n \geq 0} \frac{F_n}{n!} x^n \]

\[ = F'(x) + F(x) \]

This can be solved using ODE approach for FPS (same as if \( F(x) \) a real (or complex) function).

\[ F(x) = c_1 e^{R_+ x} + c_2 e^{R_- x} \]

where 1= \( F_0 = F(0) = c_1 + c_2 \)

\[ 1 = F_1 = F'(0) = c_1 R_+ + c_2 R_- \]
Solve for $c_1, c_2$:

$$c_1 = \frac{1}{\sqrt{5}} R_+ \quad c_2 = -\frac{1}{\sqrt{5}} R_-$$

∴ $F(x) = \frac{1}{\sqrt{5}} (R_+ e^{R_+ x} - R_- e^{R_- x})$

∴ $\sum_{n \geq 0} \frac{F_n}{n!} x^n = \frac{1}{\sqrt{5}} \sum_{n \geq 0} \frac{1}{n!} (R_+^{n+1} - R_-^{n+1}) x^n$

**DERANGEMENT RECURSION**

$$D_n = \text{number of perms in } S_n \text{ with no fixed points.}$$

$$D_{n+1} = n (D_n + D_{n+1}) \quad D_1 = 0, \quad D_2 = 1$$

**LINEAR BUT NOT CONSTANT COEFF.**

Define $D_0 = 1$ to make recursion hold for $n = 1$.

Let $D(x) = \sum_{n \geq 0} \frac{D_n}{n!} x^n$

$$= 1 + \sum_{n \geq 2} \frac{D_n}{n!} x^n$$

$$= 1 + \sum_{n \geq 0} \frac{D_{n+2}}{(n+2)!} x^{n+2}$$

∴ $D'(x) = \sum_{n \geq 0} \frac{D_{n+2}}{(n+1)!} x^{n+1}$

$$= \sum_{n \geq 0} \frac{D_{n+1} + D_n}{n!} x^{n+1}$$

$$= \sum_{n \geq 0} \frac{D_{n+1}}{n!} x^{n+1} + x D(x)$
Doesn’t look too promising? What now?

\[ \frac{1}{n!} x^n \sum_{n \geq 0} \frac{D_{n+1}}{n!} x^n + xD(x) \]

\[ D'(x) = x(D(x) - D_0)' + xD(x) \]

\[ = xD'(x) + xD(x) \]

\[ \therefore D'(x) (1 - x) = xD(x) \]

\[ \therefore \frac{D'(x)}{D(x)} = \frac{x}{1-x} = \frac{1}{1-x} - 1 \]

\[ \therefore \ln D(x) = \ln(1 - x) - x + c \]

\[ D(x) = \frac{1}{1-x} e^{-x} e^c \]

But \( D(0) = D_0 = 1 \Rightarrow e^c = 1 (\Rightarrow c = 0.) \)

\[ \therefore D(x) = \frac{e^{-x}}{1-x} \]

If you don’t know ODE, there is another way to get this result: (See Roberts, pp. 224 ff)

\[ D(x) = 1 + \sum_{n \geq 0} \frac{D_{n+2}}{(n+2)!} x^{n+2} \]

\[ D_{n+2} = (n + 1) (D_{n+1} + D_n) \]

If only this were \((n + 2)\) instead!? We can show (see Roberts, p.225)

\[ D_{n+2} = (n + 2) D_{n+1} + (-1)^{n+2} \]
\[ D(x) = 1 + \sum_{n \geq 0} \frac{D_{n+1}}{(n+1)!} x^{n+2} + \sum_{n \geq 0} \frac{(-1)^{n+2} x^{n+2}}{(n+2)!} \]

\[ = 1 + x [D(x) - 1] + (e^x - 1 + x) \]

\[ D(x) = 1 + x D(x) - x + e^x - 1 + x. \]

\[ = x D(x) + e^x \]

\[ D(x) = e^x/(1 - x) \]

**Counting Bracketings in Products**

In how many different ways can the “product” \( x_1 x_2 \ldots x_n \) be parenthesized?

**Example**

\[
\begin{align*}
(x_1) &\quad 1 \\
(x_1 x_2) &\quad 1 \\
((x_1 x_2) x_3) &\quad 2 \\
(x_1 (x_2 x_3)) &\quad \\
\end{align*}
\]

Let the number be \( b_n \). Then \( b_1 = 1, b_2 = 1, b_3 = 2 \). To bracket \( n \) letters, bracket first \( r \), last \( n - r \)

\[ b_n = \sum_{r=1}^{n-1} b_r b_{n-r} \quad , \quad n \geq 2 \]

Let \( b_0 = 0 \). Then

\[ b_n = \sum_{r=0}^{n} b_r b_{n-r} \quad , \quad n \geq 2 \]

Let \( B(x) = \sum_{n \geq 0} b_n x^n \)

\[ (B(x))^2 = \sum_{n \geq 0} c_n x^n \quad , \quad c_n = \sum_{r \geq 0} b_r b_{n-r} \]

\[ = \sum_{n \geq 2} b_n x^n \]

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\[ = B(x) - x \]

\[ \therefore B(x)^2 - B(x) + x = 0. \]

\[ B(x) = \frac{1 \pm \sqrt{1 - 4x}}{2} \]

Two possible solutions - must check each

\[ \sqrt{1 - 4x} = \sum_{n \geq 0} \binom{\frac{1}{2}}{n} (-4)^n x^n \]

Show \( \binom{\frac{1}{2}}{n} (-4)^n = -\frac{2}{n} \binom{2n-2}{n-1}, \ n \geq 1 \)

\[ \frac{1}{2} \sqrt{1 - 4x} = \frac{1}{2} - \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1} x^n \]

\[ \therefore B(x) = \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1} x^n \text{ (-ive root req'd!!)} \]

\[ \therefore b_n = \frac{1}{n} \binom{2n-2}{n-1} \text{ Catalan (1814-94)} \]

If we take the positive root we get

\[ B(x) = 1 - \sum_{n \geq 1} \frac{1}{n} \binom{2n-2}{n-1} x^n \]

which give only negative values for the coefficients for \( n \geq 1 \), which makes no sense.

A variety of problems lead to essentially the same recurrence as the above one:

(1) counting the number of simple, ordered rooted (SOR) trees

- unlabeled rooted trees, each vertex has 0, 1, or 2 descendents, “left” and “right” descendents distinguished
(2) Secondary structure in RNA [not precisely but similar - see Roberts]

(3) Triangulation of an n-gon by diagonals - division of the inside into triangles using only non-intersecting diagonals

(4) Let $S_n$ be the number of distinct ordered sets of $n$ integers $a_1, a_2, ..., a_n$ (allow some to be 0) such that
\[ a_1 + \ldots + a_n = n, \quad a_1 + a_2 + \ldots + a_k \geq k \text{ for each } k < n. \]
Then $S_n = \frac{1}{n+1} \binom{2n}{n}$

(5) Let $S_n$ be the no. of sequences of length $2n$,
\[ a_1, a_2, ..., a_{2n}, a_i = +1 \text{ or } -1 \text{ and } \]
\[ \sum_{j=1}^{2n} a_j = 0, \quad \sum_{j=1}^{k} a_j \geq 0, \quad k < 2n \]
Then $S_n = \frac{1}{n+1} \binom{2n}{n}$

**INCLUSION - EXCLUSION**

How many positive integers between 1 and 30 are not divisible by 2 or 3?
$6 = 2 \times 3$.  

Divisible by 6 $\iff$ divisible by 2 and 3.  
Exactly 15 are divisible by 2  
Exactly 10 are divisible by 3.  
Exactly 5 are divisible by 6 (hence are divisible by both 2 and 3)

$\therefore 30 - (10 + 15) + 5 = 10$ are not divisible by 2 and 3

$\{1, 5, 7, 11, 13, 17, 19, 23, 25, 29\}$

Let A be a set of N objects.

Let $a_1, a_2, \ldots, a_r$ to be collection of r properties that each of the objects of A may have (but need not have)

Let $N(a_i) = \#$ of objects of A with property $a_i$

Let $N(a'_i) = \#$ of objects of A without property $a_i$

For all $i$, $N(a_i) + N(a'_i) = N$

Let $N(a_i a_j) = \#$ of objects with both properties $a_i, a_j$

Obvious definition for $N(a_i a'_j), N(a_i a_j a_k)$ and so on

$N(a_i) = N(a_i a_j) + N(a_i a'_j)$
\[ N(a', a'_j) = N - (N(a_i) + N(a_j)) + N(a_a_j) \]

Remove object objects with add back in objects
with \(a_i\) or \(a_j\) included with both \(a_i\) and
in this count \(a_j\) since these were
removed twice

If we use above notation:

\[ N(a'_1 a'_2 ... a'_r) = N - \sum_{i} N(a_i) + \sum_{i \neq j} N(a_i a_j) - \sum_{i,j,k} N(a_i a_j a_k) + ... + (-1)^r N(a_1 a_2 ... a_r) \]

\(\text{different}\)

Proof: LHS counts each object without \(a_1\), \(a_2\), \(a_r\) exactly once. Show that the RHS does as well, and all others zero times.

- if an object \(\beta\) has none of the properties it is counted in \(N\) built never in any other term in RHS so we’re OK.
- if an object \(\beta\) has exactly \(p\) of the properties then it is counted once in \(N, \binom{p}{1} = p\) times in

\[ \sum_{i \neq j} N(a_i a_j), \binom{p}{2} \text{ times in} \sum_{i \neq j \neq k} N(a_i a_j a_k) \text{ and so on.} \]

Thus it is counted \(\sum_{j=0}^{p} \binom{p}{j} (-1)^j = 0\) times, as required

This is called the Principle of Inclusion-Exclusion (PIE)

**Corollary** The number of elements of \(A\) that have at least one of the properties is \(N - N(a'_1 a'_2 ... a'_r)\).

**Exercise 1** Find the number of non negative integers satisfying solutions to \(x_1 + x_2 + x_3 + x_4 = 18\) with each \(x_i \leq 7\).

Solution: Without the restriction on all \(x_i\) the answer is

\[ \binom{4 + 18 - 1}{18} = \binom{21}{18} = \binom{21}{3} \]

Define the property \(a_i\) for a solution to the equation if \(x_i \geq 8\), i.e. a solution \((x_1, x_2, x_3, x_4)\) satisfies \(a_i\) if \(x_i \geq 8\), \(i = 1, 2, 3, 4\). We want to count