b. done in the lecture notes on derangements.

c. \[ D_0 = 1 \quad D_1 = 0 \quad D_2 = 1 \quad D_3 = 2 \]
\[ D_4 = 9 \quad D_5 = 44 \quad D_6 = 265 \]
\[ D_7 = 1854 \quad D_8 = 14,853 \]
\[ D_9 = 133,496 \quad D_{10} = 1,334,961 \]

d. Let \( D(x) = \sum_{n=0}^{\infty} \frac{D_n x^n}{n!} \)

\[ D'(x) = \sum_{n=1}^{\infty} \frac{D_n x^{n-1}}{(n-1)!} \]
\[ = \sum_{n=1}^{\infty} \frac{(n-1) (D_{n-1} + D_{n-2}) x^{n-1}}{(n-1)!} \]

\[ D''(x) = \sum_{n=1}^{\infty} \frac{(n-1) D_{n-1} x^{n-1}}{(n-1)!} + \sum_{n=2}^{\infty} \frac{n D_{n-2} x^{n-1}}{(n-1)!} \]

\[ D'''(x) = x \cdot \sum_{n=1}^{\infty} \frac{D_n x^{n-2}}{(n-2)!} + x \cdot \sum_{n=2}^{\infty} \frac{D_{n-2} x^{n-2}}{(n-2)!} \]
\[ D'(x) = x \cdot D(x) + x^2 \cdot 1 \cdot x \cdot D(x) \]

\[ D'(x) \frac{1}{1-x} = x \cdot D(x) \]

\[ \frac{D(x)}{1-x} = -1 + \frac{1}{1-x} \]

\[ \int \frac{D(x)}{1-x} \, dx = \int \left( -1 + \frac{1}{1-x} \right) \, dx \]

\[ \ln |1-x| = -x + \ln |1-x| + C \]

\[ D(x) = C e^{-x} \cdot \frac{1}{(1-x)} \]

\[ D(0) = D_0 = 1 \Rightarrow C_1 = 1 \]

\[ D(x) = e^{-x} \cdot (1-x)^{-1} \]

37. INTELLIGENT

II TO BE INGENIOUS

Let \( U \) be the universal set of all arrangements of the word intelligent with no restrictions.

The number of arrangements with at least 2 pairs of consecutive identical letters

\[ N = N_1 + N_2 \]

Arrangements with no restrictions.

Number of arrangements with no pairs of consecutive identical letters.

Number of arrangements with exactly one pair of consecutive identical letters.
\[ \mathcal{N} = \frac{11!}{(2!)^3} \]

For \( \mathcal{N}_1 \),

Let \( A_i \) = set of arrangements where TT together.

\[
\begin{align*}
A_1^1 & = "TT" \\
A_2^1 & = "TT" \\
A_3^1 & = "TT" \\
A_4^1 & = "TT" \\
A_5^1 & = "TT" \\
A_6^1 & = "TT" \\
A_7^1 & = "TT" \\
\end{align*}
\]

\[ \mathcal{N}(A_1^1) = \text{put } TT \text{ together.} \]

\[
\mathcal{N}(A_2^1) = \mathcal{N}(A_3^1) = \mathcal{N}(A_4^1) = \mathcal{N}(A_5^1) = \mathcal{N}(A_6^1) = \mathcal{N}(A_7^1) = \mathcal{N}(A_i) = \frac{10!}{(2!)^3}\]

\[ \mathcal{N}(A_i; A_j) = \frac{9!}{(2!)^3} \left[ \begin{array}{c} \text{put } TT, \text{ other } \text{ pairs together.} \\
\text{arrangements with } \frac{9!}{(2!)^3} \text{ TT together} \\
\text{arrangement is the same with any other } \text{ pairs together.} \end{array} \right] \]

\[ \mathcal{N}(A_i; A_j; A_k) = \frac{8!}{(2!)^2} \left[ \begin{array}{c} \text{put } \text{ other } \text{ pairs together}. \\
\frac{8!}{(2!)^2} \text{ arrangements with the other pairs together.} \end{array} \right] \]

\[ \mathcal{N}(A_i; A_j; A_k; A_\ell) = \frac{7!}{2!} \]

\[ \mathcal{N}(A_1 A_2 A_3 A_4 A_5) = 6! \]

\[ \mathcal{N}(A_1 A_2 A_3 A_4 A_5) = \]

\[ \mathcal{N}(A_1 A_2 A_3 A_4 A_5) = \frac{11!}{(2!)^3} - \left( \begin{array}{c} 5 \\ 1 \end{array} \right) \frac{10!}{2!} + \left( \begin{array}{c} 5 \\ 2 \end{array} \right) \frac{9!}{2!} - \left( \begin{array}{c} 5 \\ 3 \end{array} \right) \frac{6!}{2!} + \left( \begin{array}{c} 5 \\ 4 \end{array} \right) \frac{7!}{2!} = 6! \]
For $N_y$, there exists one year

\[ N_1 = N(\overline{A_1 A_2 A_3 A_4 A_5}) + N(\overline{A_2 A_3 A_4 A_5 A_1}) + N(\overline{A_3 A_4 A_5 A_1 A_2}) + N(\overline{A_4 A_5 A_1 A_2 A_3}) + N(\overline{A_5 A_1 A_2 A_3 A_4}) \]

Those will, in total, be 252. However, since there are two letters of each type,

\[ N(\overline{A_1 A_2 A_3 A_4}) = N(\overline{A_1 A_2 A_3 A_4}) \]

\[ = N(A_1) - N(A_1 A_6 A_5) - N(A_1 A_6 A_5) \]

\[ = N(A_1) - \left[ N(A_1 A_6 A_5) + N(A_1 A_6 A_5) + N(A_1 A_6 A_5) + N(A_1 A_6 A_5) - N(A_1 A_6 A_5) - N(A_1 A_6 A_5) - N(A_1 A_6 A_5) - N(A_1 A_6 A_5) \right] \]

\[ = \frac{101}{(21)^4} \left[ 4 \cdot \frac{91}{(21)^3} - 6 \cdot \frac{81}{(21)^2} + 4 \cdot \frac{71}{2!} - 6 \right] \]

\[ N_2 = \frac{5 \cdot 101}{(42)^4} - \left[ \frac{20 \cdot 91}{(42)^3} - \frac{30 \cdot 81}{(42)^2} + \frac{20 \cdot 71}{2!} - 5 \cdot 6 \right] \]

\[ N_y = N_1 - N_2 \]

\[ = \frac{11}{(21)^5} - \frac{11}{(21)^5} + \frac{5}{(21)^4} + \frac{6}{(21)^3} + \frac{5}{(21)^2} + \frac{5}{(21)} \cdot \frac{6}{2!} - \frac{5}{(21)} + \frac{5}{2!} + \frac{5}{2!} + \frac{5}{2!} + 5 \cdot 6 \]

\[ = \frac{10 \cdot 5}{(21)^3} - \frac{20 \cdot 5}{(21)^2} + \frac{10 \cdot 5}{2!} - 5 \cdot 6 \]
The question should read

Prove that
\[ S_m = \sum_{k=m}^{n} \binom{n}{k} N_k \]

Recall that
\[ N_m = \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} S_k \]

RHS of (k)
\[ = N_n + \binom{m^1}{m} N_{m^1} + \binom{m^2}{m} N_{m^2} + \ldots + \binom{n}{m} N_n \]

Since \( N_m, N_{m^1}, N_{m^2}, \ldots, N_n \)
contain \( S_{m}, S_{m^1}, S_{m^2}, \ldots, S_{n} \), our objective is to show
that the coefficient of \( S_{m^1}, S_{m^2}, \ldots, S_{n} \)
is 0.

\[ \sum_{k=m}^{n} (-1)^{k-m} \binom{k}{m} S_k + \sum_{k=m+1}^{n} (-1)^{k-m-1} \binom{k}{m+1} S_k + \ldots + \sum_{k=n}^{n} (-1)^{n-k} \binom{n}{n} S_k \]

\[ = S_m + c_{m^1} S_{m^1} + \ldots + c_{m^2} S_{m^2} + \ldots + c_{n} S_{n} \]

The term with \( S_k, k \geq m+1 \) is

\[ (-1)^{k-m} \binom{k}{m} S_k + (-1)^{k-m-1} \binom{k}{m+1} S_k + (-1)^{k-m-2} \binom{k}{m+2} S_k + \ldots \]

\[ = (-1)^{k-m} S_k \left[ \binom{k}{m} - \binom{m^1}{m} \binom{k}{m^1} + \binom{m^2}{m} \binom{k}{m^2} - \binom{m^3}{m} \binom{k}{m^3} + \ldots \right] \]

\[ \binom{m+1}{m} \binom{k}{m} = \frac{(m+1)!}{m! \cdot 1!} = \binom{k}{k-m} = \frac{k!}{(k-m-1)!} = \frac{k!}{m! \cdot (k-m)!} \cdot \frac{1}{m! \cdot (k-m-1)!} \]

\[ = \binom{k}{m} - \binom{m^1}{m} \binom{k}{m^1} + \binom{m^2}{m} \binom{k}{m^2} - \binom{m^3}{m} \binom{k}{m^3} + \ldots \]
\[
\binom{m+2}{m} \binom{k}{m} = \binom{m+2}{m} \frac{k!}{\omega! (k-m-1)! m! (k-m)!} = \frac{k!}{m! (k-m)!} \binom{k-m}{2} = \binom{k}{m} \binom{k-m}{2}
\]

It's not hard to see that
\[
\binom{m+2}{m} \binom{k}{m+2} = \binom{k}{m} \binom{k-m}{2}
\]

Hence the coefficient of \( S_k \) can be expressed as
\[
(-1)^{k-m} \left[ (k) - (k-m)(k) + (k-m)(k) - (k-m)(k) + \cdots - (k-m) \binom{k}{m} \right]
\]
\[
= (-1)^{k-m} \left[ 1 - \binom{k-m}{1} + \binom{k-m}{2} - \binom{k-m}{3} + \cdots - 1 \right]
\]
\[
= (-1)^{k-m} (1 - 1)^{k-m} = 0
\]

This shows that \( c_k = 0 \) for \( k \geq m+1 \).

\[
\sum_{k=m}^{n} \binom{k}{m} N_k = S_m + 0 \cdot S_{m+1} + 0 \cdot S_{m+2} + \cdots + 0 \cdot S_n = S_m.
\]
13. Use a Venn Diagram

\[
\begin{array}{ccc}
\text{Trigonometry} & \text{Probability} & \text{Basket weaving} \\
35 & 35 & 35 \\
15 & 15 & 15 \\
35 & & \\
\end{array}
\]

Fill in the numbers

a. 35 are taking none of these subjects.
b. 35 are only taking probability.

14. Let's say there are 100 college professors at the institution where these results were obtained. Use a Venn Diagram

\[
\begin{array}{ccc}
\text{like Tennis} & \text{like Chess} & \text{like bridge} \\
25 & 25 & 25 \\
25 & 25 & \\
\end{array}
\]

a. fill in 20 where all three circles intersect.

Filling the other information, we are forced to conclude that at least 70 out of the 100 college professors like tennis. However, we were told that only 60 like tennis. We should be suspicious of these results.
6. Fill in $x$ where all three circles intersect.

Now $y + 45 - x + x + 45 - x = 60$, $z + 90 - 2y + x = 50$

$y - 90 - x = 0$

$2 - x = -45$

$y - x = -30$

$x - 2 = 40$  (2)

i.e. $x - y = 30$  (1)

$w + 90 - 2x + x = 65$

$w - x = -35$  (3)

$X - w = 35$

$x - y = 30$  (1)

$x - 2 = 40$  (2)

$x - w = 35$  (3)

Using (2) $x$ must be at least 40.

Smallest percentage would be 40%.
21. Let \( A_1 = \) the set of arrangements of TAMELY with \( T \) before \( A \).
   Let \( A_2 = \) the set of arrangements of TAMELY with \( A \) before \( M \).
   Let \( A_3 = \) the set of arrangements of TAMELY with \( M \) before \( E \).

   We want \( N(A_1 A_2 A_3) \).

   \[
   N(A_1 A_2 A_3) = N(A_1) + N(A_2) + N(A_3) - N(A_1 A_2) - N(A_1 A_3) - N(A_2 A_3) + N(A_1 A_2 A_3)
   \]

   For \( N(A_1) \),
   choose 2 positions out of the six available for the \( T \) and \( A \).
   There are \( \binom{6}{2} \) ways to do this. Once these positions are selected, there is only one way (i.e., one order) in which we can place the \( T \) and the \( A \).
   The other four letters can be arranged in \( 4! \) ways.

   \[
   N(A_1) = \binom{6}{2} \times 4!
   \]

   For \( N(A_2) \) and \( N(A_3) \), the argument is identical to the argument used for \( N(A_1) \).

   Hence \( N(A_2) = N(A_3) = \binom{6}{2} \times 4! \)

   For \( N(A_1 A_2) \),
   choose 3 of these positions out of 6 for \( T, A, M \).
   There are \( \binom{6}{3} \) ways to do this.
Once the positions are selected, there is only one order in which T, A, H can be placed in. The remaining 3 letters can be arranged in 3! ways.

\[ N(A, nA_2) = \binom{5}{3} \times 3! \]

The argument for \( N(A_2 nA_3) \) is identical to the one used for \( N(A, nA_2) \).

\[ N(A_2 nA_3) = \binom{6}{3} \times 3! \]

For \( N(A, nA_3) \), choose 4 positions for T, A, M, E. This can be done in \( \binom{6}{4} \) ways. Out of these 4 positions choose 2 for T, A, i.e. \( \binom{4}{2} \) ways. Once the positions are selected there is only one order in which T, A can be placed in. The 2 remaining letters can be arranged 2! ways.

\[ N(A, nA_3) = \binom{6}{4} \binom{4}{2} \times 2! \]

For \( N(A, nA_2 nA_3) \), TAME must be together, so choose 4 positions out of six for these letters. There is only one order they can be placed in once the positions are selected. The remaining 2 letters can be arranged 2! ways.

\[ N(A, nA_2 nA_3) = \binom{6}{4} \times 2! \]
\[ N(A_1 \cup A_2 \cup A_3) = 3 \times \binom{6}{2} \times 4! - \left[ 2 \times \binom{6}{3} \times 3! + \binom{6}{4} \binom{4}{2} \times 2! \right] + \binom{6}{4} \times 2! \]

22. Let \( A_1 \) = the set of arrangements of \( \text{MATHEMATICS} \) with both \( T \)'s before both \( A \)'s.

Let \( A_2 \) = the set of arrangements of \( \text{MATHEMATICS} \) with both \( A \)'s before both \( M \)'s.

Let \( A_3 \) = the set of arrangements of \( \text{MATHEMATICS} \) with both \( M \)'s before \( E \).

We want \( N(A_1 \cup A_2 \cup A_3) \).

Use the same formula as in question 21.

\[ N(A_1) = \binom{11}{4} \frac{7!}{2!} \]

11 positions
Choose four for \( T, T, A, A \)

Arrange the rest of the letters.

\[ N(A_2) = \binom{11}{4} \frac{7!}{2!} \]

11 positions
Choose four positions for \( M, M, A, A \)

Arrange the rest of the letters.

in both cases once the positions are selected, there is only one way to place the letters.
\[ N(A_3) = \binom{11}{3} \times \frac{8!}{2!3!} \]

- There is only one way to place the letters after the positions are chosen.

\[ N(A_1 n A_2) = \binom{11}{6} \times 5! \]

\[ N(A_2 n A_3) = \binom{11}{5} \times \frac{6!}{2!} \]

\[ N(A_1 n A_3) = \binom{11}{7} \binom{7}{4} \times 4! \]

\[ N(A_1 n A_2 n A_3) = \binom{11}{7} \times 4! \]

Choose 7 for \(TAAMME\) and the other letters.

Choose 7 for \(T, T, A, A, M, M, E\). They must go in as \(TTAAMME\).
\[ N(A, UA_2, UA_3) = 2 \times \binom{11}{4} \times \frac{7!}{2!} + \binom{11}{3} \times \frac{8!}{2!2!} - \binom{11}{6} \times 5! - \binom{11}{5} \times \frac{6!}{2!} - \binom{11}{7} \frac{4!}{4} + \binom{11}{7} \times 4! \]