We could repeat the same argument for $b_n$. If the first digit is a 0 or 2, then we are left with an $n-1$-digit quaternary sequence requiring an odd total number of 0's and 1's. In each case we get $b_{n-1}$ for a total of $2b_{n-1}$.

If the first digit is 0 or 1, then we are left with an $n-1$-digit quaternary sequence requiring an even total number of 0's and 1's. In each case we get $a_{n-1}$ for a total of $2a_{n-1}$. Hence $b_n = 2b_{n-1} + 2a_{n-1}$.

C. Let $a_n$ be the number of quaternary sequences with an even number of 0's and an even number of 1's. If the first digit is a 0 or 3, then we are left with an $n-1$-digit quaternary sequence requiring an even number of 0's and an even number of 1's. In each case we get $a_{n-1}$ for a total of $2a_{n-1}$.

Let $b_n$ be the number of quaternary sequences with an odd number of 0's and an even number of 1's. Let $c_n$ be the number of quaternary sequences with an even number of 0's and an odd number of 1's. If the first digit is 0, then we are left with an $n-1$-digit quaternary sequence requiring an odd number of 0's and an even number of 1's. There are $b_{n-1}$ of these. If the first digit is 3, then we are left with an $n-1$-digit quaternary sequence requiring an odd number of 1's and an even number of 0's. There are $c_{n-1}$ of these.
Hence \( A_n = 2A_{n-1} + b_{n-1} + c_{n-1} \)

We need to develop a recurrence relation for \( b_n \).

If the first digit is a 2 or 3, then we are left with an \( n-1 \) digit quaternary sequence requiring an odd number of 0's and an even number of 1's. In each case we get \( b_{n-1} \), for a total of \( 2b_{n-1} \).

If the first digit is a 0, then we are left with an \( n-1 \) digit quaternary sequence requiring an even number of 0's and an even number of 1's. There are \( a_{n-1} \) of these.

If the first digit is a 1, then we are left with an \( n-1 \) digit quaternary sequence with an odd number of 1's and an odd number of 0's. We don't have anything for this type so we will consider subcases via the second digit.

If the second digit is 0, then we are left with an \( n-2 \) digit sequence requiring an even number of 0's and an odd number of 1's. There are \( c_{n-2} \) of these.

If the second digit is 1, then we are left with an \( n-2 \) digit sequence with an even number of 1's and an odd number of 0's. There are \( b_{n-2} \) of these.

Hence \( b_n = 2b_{n-1} + a_{n-1} + c_{n-2} + b_{n-2} \).
We need to develop a system of recurrence relations for \( C_n \).

If the first digit is a 2 or 3, then we are left with an \( n-1 \) digit quaternary sequence requiring an even number of 0's and an odd number of 1's. In each case we get \( C_{n-1} \) for a total of \( 2C_{n-1} \).

If the first digit is a 1, then we are left with an \( n-1 \) digit quaternary sequence requiring an even number of 1's and an even number of 0's. There are \( 0_{n-1} \) of these.

If the first digit is a 0, then we are left with an \( n-1 \) digit quaternary sequence requiring an odd number of 0's and an odd number of 1's. We don't have anything for this type so we will consider subcases via the second digit.

If the second digit is a 0, then we are left with an \( n-2 \) digit quaternary sequence requiring an even number of 0's and an odd number of 1's. There are \( C_{n-2} \) of these.

If the second digit is a 1, then we are left with an \( n-2 \) digit quaternary sequence requiring an odd number of 0's and an even number of 1's. There are \( b_{n-2} \) of these.

Hence \( C_n = 2C_{n-1} + 0_{n-1} + C_{n-2} + b_{n-2} \).
41 b) Show that \[ \sum_{k=0}^{n/2} f(n,k) = F_n, \]

where \( F_n \) is the \( n \)th Fibonacci number.

Since \( n \) is even, let \( n = 2m. \)

We will show that \[ \sum_{k=0}^{m} f(2m,k) = F_{2m+1} \]

We need a result concerning \( F_{2m+1} \).

\[ F_{2m+1} = 2F_{2m-1} + F_{2m-3} + F_{2m-5} + \ldots + F_1 + F_0 \]

Proof by Induction (Strong)

Base case: \( m = 1 \)

LHS \[ = \sum_{k=0}^{1} f(2, k) = f(2, 0) + f(2, 1) \]

\[ \begin{align*}
= 1 + 1 \\
= 2
\end{align*} \]

RHS \[ = F_2 \]

\[ = F_2 + F_1 \]

\[ = 3 \]

Suppose that \( m = 0, 1, 2, \ldots, n \)

\[ \sum_{k=0}^{m} f(2m, k) = F_{2m+1} \]

We want to prove that \[ \sum_{k=0}^{n+1} f(2m+2, k) = F_{2m+3} \]

\[ \sum_{k=0}^{n+1} f(2m+2, k) = \sum_{k=0}^{m+1} f(2m+1, k) + \sum_{k=0}^{n+1} f(2m, k-1) \]

\[ \begin{align*}
= \sum_{k=0}^{m+1} f(2m+1, k) + \sum_{k=1}^{n+1} f(2m, k-1) \\
\end{align*} \]

Since \( f(2m, n+1) = 0 \)

\[ \begin{align*}
= \sum_{k=0}^{m+1} f(2m+1, k) + \sum_{j=0}^{n} f(2m, j) \\
\end{align*} \]

\[ \begin{align*}
= \sum_{k=0}^{m+1} f(2m+1, k) + F_{2m+1} \\
\end{align*} \]
\[
= \sum_{k=0}^{n} f(2n, k) + \sum_{k=0}^{n} f(2n-1, k-1) + F_{2n+1}
\]

Using I.H.
\[
= 2F_{2n+1} + \sum_{k=1}^{n} f(2n-1, k-1)
\]

Using I.H.
\[
= 2F_{2n+1} + F_{2n-1} + \sum_{k=2}^{n} f(2n-2, k-2)
\]

Using I.H.
\[
= 2F_{2n+1} + F_{2n-1} + \sum_{k=2}^{n} f(2n-5, k-3)
\]

Using I.H.
\[
= 2F_{2n+1} + F_{2n-1} + F_{2n-3} + \sum_{k=3}^{n} f(2n-5, k-3)
\]

Using I.H.
\[
= 2F_{2n+1} + F_{2n-1} + F_{2n-3} + F_{2n-5} + \sum_{k=4}^{n} f(2n-7, k-4)
\]

Continuing this process, we will get to
\[
2F_{2n+1} + F_{2n-1} + F_{2n-3} + F_{2n-5} + \ldots + F_1 + F_0.
\]

Reason:
\[
\sum_{k=j}^{n} f(\frac{2n-(2j-1)}{k-j}) = \sum_{k=j}^{n} f(\frac{2n-2j}{k-j}) + \sum_{k=j}^{n} f(\frac{2n-2j+1}{k-j-1})
\]

\[
= F_{2n-2j+1} + \sum_{k=j}^{n} f(\frac{2n-2j-1}{k-j-1}).
\]
Section 7.3 Exercises

3.

a. \( a_n = 3a_{n-1} + 4a_{n-2} \quad a_0 = a_1 = 1 \)

\[ a_n = \alpha^n \]

\[ \alpha^n = 3\alpha^{n-1} + 4\alpha^{n-2} \]

\[ \alpha^2 = 3\alpha + 4 \]

\[ \alpha^2 - 3\alpha - 4 = 0 \]

\[ (\alpha - 4)(\alpha + 1) = 0 \]

\[ \alpha = 4 \quad \alpha = -1 \]

General solution \( a_n = A_1(4)^n + A_2(-1)^n \)

\[ a_0 = a_1 + A_2 = 1 \]

\[ a_1 = 4A_1 - A_2 = 1 \]

\[ A_1 = \frac{2}{5} \quad A_2 = \frac{3}{5} \]

\[ a_n = \frac{2}{5}(4)^n + \frac{3}{5}(-1)^n \]

b. \( a_n = a_{n-2} \quad a_0 = a_1 = 1 \)

\[ a_n = \alpha^n \]

\[ \alpha^n = \alpha^{n-2} \]

\[ \alpha^2 = 1 \]

\[ \alpha = \pm 1 \]

General solution \( a_n = A_1(1)^n + A_2(-1)^n = A_1 + A_2(-1)^n \)

\[ a_0 = A_1 + A_2 = 1 \]

\[ a_1 = A_1 - A_2 = -1 \]

\[ A_1 = 1 \quad A_2 = 0 \]

\[ a_n = 1(1)^n = 1 \]

c. \( a_n = 2a_{n-1} - a_{n-2} \quad a_0 = a_1 = 2 \)

\[ a_n = \alpha^n \]

\[ \alpha^n = 2\alpha^{n-1} - \alpha^{n-2} \]

\[ \alpha^2 = 2\alpha - 1 \]

\[ \alpha^2 - 2\alpha + 1 = 0 \]

\[ (\alpha - 1)^2 = 0 \]

\[ \alpha = 1, 1 \quad \text{double root} \]

General solution \( a_n = A_1(1)^n + A_2n(1)^n = A_1 + nA_2 \)

\[ a_0 = A_1 + 0 = 2 \]

\[ a_1 = 2 + A_2 = 2 \]

\[ A_2 = 0 \]

\[ a_n = 2 \]
1. \( a_n - 3a_{n-1} - 3a_{n-2} + a_{n-3} = 0 \)  \( n \geq 3 \)  \( a_2 = 2 \)

\[ a_n = 2^n \]

\[ a^3 - 3a^2 + 3a - 1 = 0 \]

\[ (a-1)^3 = 0 \]

\[ a = 1, 1, \text{ repeated roots.} \]

General solution

\[ a_n = A_1 (1)^n + A_2 n (1)^n + A_3 n^2 (1)^n \]

\[ a_0 = A_1 = 1 \]

\[ a_1 = 1 + A_2 + A_3 = 1 \]

\[ 2A_2 + 3A_3 = 0 \]

\[ A_2 = 1 + 2A_2 + 3A_3 = 2 \]

\[ 4A_2 + 2A_3 = 1 \]

\[ A_2 + A_3 = 0 \]

\[ A_3 = \frac{1}{2}, A_2 = -\frac{1}{2} \]

\[ a_n = \frac{1}{2} n^2 - \frac{1}{2} n + 1 \]

2. \( b_n - b_{n-1} = 2 (b_{n-1} - b_{n-2}) \)

This year change

\[ a_n - a_{n-1} = 2 (a_{n-1} - a_{n-2}) \]

\[ a^2 - 2a = 8a - 2 \]

\[ a^2 - 3a + 2 = 0 \]

\[ (a-1)(a-2) = 0 \]

\[ a = 2 \text{ or } a = 1 \]

General solution \( b_n = A_1 (2)^n + A_2 (1)^n \)

\[ b_0 = A_1 + A_2 = 1 \]

\[ b_1 = 2A_1 + A_2 = 1 \]

\[ A_1 = 3 \]

\[ A_2 = -2 \]

\[ b_n = 3 (2)^n - 2 (1)^n \]

\[ = 3 - 2^n - 2 \]
10. **Fibonacci Relation**

\[ F_n = F_{n-1} + F_{n-2} \quad F_0 = F_1 = 1 \]

\[ F_n = \alpha^n \]

\[ \alpha^n = \alpha^{n-1} + \alpha^{n-2} \]

\[ \alpha^2 = \alpha + 1 \]

\[ \alpha^2 - \alpha - 1 = 0 \]

\[ \alpha = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \]

**General Solution**

\[ F_n = A_1 \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^n + A_2 \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^n \]

\[ F_0 = F_1 = 1 \]

\[ \Rightarrow A_1 = \frac{1}{\sqrt{5}} \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right) \]

\[ A_2 = \frac{1}{\sqrt{5}} \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right) \]

\[ F_n = \frac{1}{\sqrt{5}} \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{n+1} + \frac{1}{\sqrt{5}} \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{n+1} \]

\[ \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{\frac{1}{\sqrt{5}} \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{n+2} + \frac{1}{\sqrt{5}} \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{n+2}}{\frac{1}{\sqrt{5}} \left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{n+1} + \frac{1}{\sqrt{5}} \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{n+1}} \]

\[ \left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right) < 0 \]

\[ \lim_{n \to \infty} \left( \frac{\left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{n+2}}{\left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{n+1}} + \frac{\left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{n+2}}{\left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{n+1}} \right) \]

\[ = 1 + \frac{\left( \frac{1}{2} - \frac{\sqrt{5}}{2} \right)^{n+1}}{\left( \frac{1}{2} + \frac{\sqrt{5}}{2} \right)^{n+1}} \]

\[ = \frac{1}{2} \pm \frac{\sqrt{5}}{2} \]
Section 6.4

22. \( P(x) = \sum_{k=0}^{\infty} p_k x^k \)

0. \( m_k = \sum_{j=0}^{\infty} j^k \beta_j \)

exponential generating function for \( m_k \)

\[ \sum_{k=0}^{\infty} m_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} j^k \beta_j \right) \frac{x^k}{k!} \]

\[ = p_1 + p_2 x + p_3 x^2 + \ldots \]

\[ + \left( \beta_1 + \beta_2 x + \beta_3 x^2 + \ldots \right) \frac{x^1}{1!} \]

\[ + \left( \beta_2 + \beta_3 x + \beta_4 x^2 + \ldots \right) \frac{x^2}{2!} \]

\[ + \left( \beta_3 + \beta_4 x + \beta_5 x^2 + \ldots \right) \frac{x^3}{3!} \]

\[ + \ldots \]

Change variables, fix \( j \) and let \( k \) run.

\[ p_1 + p_j x + p_j^2 x^2 + \ldots + \frac{p_j^k x^k}{k!} \]

\[ = p_j \left( 1 + jx + \frac{j^2 x^2}{2!} + \ldots + \frac{j^k x^k}{k!} \right) \]

\[ = p_j e^{jx} \]

\[ \sum_{j=0}^{\infty} \beta_j e^{jx} = P(e^x) \]
Kth factorial moment

\[ m_k = \sum_{j=0}^{\infty} \frac{j!}{(j-k)!} \rho_j \]

Expansion generating function for \( m_k \)

\[ \sum_{k=0}^{\infty} m_k \frac{x^k}{k!} = \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} \rho_j \frac{j!}{(j-k)!} \right) \frac{x^k}{k!} \]

\[ = \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} \rho_j \frac{(j)!}{(j-k)!} \right) x^k \]

\[ = \sum_{k=0}^{\infty} \left( \sum_{j=k}^{\infty} \rho_j \left( \frac{j}{k} \right) \right) x^k \]

\[ = \sum_{k=0}^{\infty} \rho_k \frac{x^k}{k!} + \rho_1 \frac{x^{k+1}}{(k+1)!} + \cdots \]

\[ + \rho_2 \frac{x^{k+2}}{(k+2)!} + \cdots \]

\[ + \rho_3 \frac{x^{k+3}}{(k+3)!} + \cdots \]

\[ + \rho_j \frac{x^{k+j}}{(k+j)!} + \cdots \]

\[ + \rho_{j+1} \frac{x^{k+j+1}}{(k+j+1)!} + \cdots \]

\[ + \cdots \]

\[ + \sum_{j=0}^{\infty} \rho_j \left( 1 + \frac{x}{j} \right)^j = \rho(x+1) \]
C. \( X \) number of heads when \( n \) coins are 

\[ P(X = k) = \binom{n}{k} n^{-k} \]

\[ g(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k e^{-x} \]

\[ m_i = g^{(i)}(0) \quad g^{(i)}(x) = \sum_{k=0}^{\infty} \frac{\binom{n}{k}}{k^i} x^k \]

\[ m_i = g^{(i)}(0) = \frac{1}{2^n} \sum_{k=0}^{\infty} k^i \binom{n}{k} = \frac{1}{2^n} \sum_{k=0}^{\infty} \binom{n}{k} k^i \]

\[ (1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k \]

\[ n(1+x)^{n-1} = \sum_{k=0}^{\infty} \binom{n}{k} x^{k-1} \]

\[ (2)^{n-1} = \sum_{k=0}^{\infty} \binom{n}{k} \]

\[ m_2 = g^{(2)}(0) \quad g^{(2)}(x) = \sum_{k=0}^{\infty} \frac{2 \binom{n}{k}}{2^n} x^k \]

\[ m_2 = g^{(2)}(0) = \sum_{k=0}^{\infty} \frac{2 \binom{n}{k}}{2^n} = \frac{1}{2^n} \sum_{k=0}^{\infty} \binom{n}{k} k^2 \]

\[ n(n-1)(1+x)^{n-2} = \sum_{k=0}^{\infty} \binom{n}{k}(k)(x)^{k-1} \]

\[ n(n-1)2 = \sum_{k=0}^{\infty} \binom{n}{k}(k)(x)^{k-1} = \sum_{k=0}^{\infty} \binom{n}{k} k - \sum_{k=0}^{\infty} \binom{n}{k} k^2 \]

\[ n(n-1)2^{n-1} = \sum_{k=0}^{\infty} \binom{n}{k} k^2 \]

\[ n(n+1)2^{n-1} = \sum_{k=0}^{\infty} \binom{n}{k} k^2 \]

\[ n(n+1)2^{n-1} = \sum_{k=0}^{\infty} \binom{n}{k} k^2 \]
\[ g(x) = \sum_{k=0}^{\infty} \frac{(x)^k}{k!} (x+1)^k \]

\[ m_2^* = g''(a) \]

\[ g''(x) = \sum_{k=0}^{\infty} \frac{(x)^{k-2} (x+1)^k}{k-2} \]

\[ \frac{1}{2^n} \sum_{k=0}^{\infty} k(k-1)(x)^k \]

\[ = \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+2} - \frac{x^k}{k} \]

\[ d_{\text{in poisson}} \]

\[ \mu = \frac{n(n-1)}{4} \]

\[ p(x=k) = \frac{\mu^k e^{-\mu}}{k!} \]

\[ m_1 = g'(a) \]

\[ g'(x) = \sum_{k=0}^{\infty} \frac{(x)^{k-1}}{k!} (x+1)^k \]

\[ g'(a) = \sum_{k=0}^{\infty} \frac{(a)^{k-1}}{k!} \frac{d}{dx} (x+1)^k \]

\[ = e^{-\mu} \sum_{k=0}^{\infty} \frac{(a)^k}{(k+1)!} = e^{-\mu} \sum_{k=0}^{\infty} \frac{(a)^{k+1}}{(k+1)!} \]

\[ m_2 = g'(a) \]

\[ g'(x) = \sum_{k=0}^{\infty} \frac{(x)^{k-1}}{k!} (x+1)^k \cdot k(x+1)^k \]

\[ m_2 = g'(a) = \sum_{k=0}^{\infty} \frac{(a)^{k-1}}{k!} k(a+1)^k = e^{-\mu} \sum_{k=0}^{\infty} \frac{(a)^{k+1}}{(k+1)!} \]

\[ = e^{-\mu} \sum_{k=0}^{\infty} \frac{(a)^{k+1}}{(k+1)!} = \mu e^{-\mu} \]

\[ = e^{-\mu} \sum_{k=0}^{\infty} \frac{(a)^{k+1}}{(k+1)!} = \mu e^{-\mu} \]
Section 6.5 # 8

\[ A(x) = \sum_{r=0}^{\infty} a_r x^r \quad S = \sum_{k=1}^{\infty} a_k \]

generating function for \( S \), \( S = \sum_{r=0}^{\infty} a_r x^r \)

\[ \sum_{r=0}^{\infty} a_r x^r - \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} a_k \]

\[ q_0 + q_0 + q_0 + \ldots + q_0 + \ldots \]

\[ + q_0 x + q_0 x + \ldots + a x + \ldots \]

\[ + q_0 x^2 + \ldots + a x^2 + \ldots \]

\[ + \]

\[ q_0 x^3 + \ldots + a x^3 + \ldots \]

\[ + \]

\[ q_0 x^4 + \ldots + a x^4 + \ldots \]

Reverse the order of summation.

\[ \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} a_j \cdot x^k = \sum_{j=1}^{\infty} a_j \sum_{k=0}^{j-1} x^k \]

\[ = \sum_{j=1}^{\infty} (a_j \cdot \frac{1-x^j}{1-x}) = \frac{1}{1-x} \sum_{j=1}^{\infty} a_j (1-x^j) \]

\[ = \frac{1}{1-x} \left( \sum_{j=1}^{\infty} a_j - \sum_{j=1}^{\infty} a_j x^j \right) \]

\[ = \frac{1}{1-x} \left( \sum_{j=1}^{\infty} a_j (1-x^j) \right) \]

\[ = \frac{1}{1-x} \left( \sum_{j=1}^{\infty} a_j x^j \right) \]

\[ = \frac{1}{1-x} \left( x \cdot h(x) \right) \]

\[ = \frac{x h(x) - x}{x-1} \]

Since \( a \) is a constant generating function \( \frac{x h(x) - x}{x-1} \)