Chapter 9: Symbolic Dynamics

9.1 Itineraries
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Chapter 10: Chaos

10.1 The Three Properties of a Chaotic System
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Points with Prime Period 3 for $Q_c(x) = x^2 + c$

Suppose we try to find exactly what are the points with prime period 3 for $Q_c(x) = x^2 + c$.

$$Q_c^3(x) = x \iff ((x^2 + c)^2 + c)^2 + c = x \iff x^2 - x + c = 0$$
or $x$ is a root of the sixth degree polynomial:

$$x^6 + x^5 + (3c+1)x^4 + (2c+1)x^3 + (3c^2+3c+1)x^2 + (c^2+2c+1)x + c^3 + 2c^2 + c + 1.$$ 

This is not easy to solve! What is needed to describe theoretically the dynamics of the quadratic family is another way to look at the whole situation. Analytic calculations are just too difficult. Enter symbolic dynamics! That is, abstract the situation in some fashion such that the calculations become easier!

Recall (or Check) the Following for $Q_c, c < -2$

1. Fixed points of $Q_c$ are $p_- = \frac{1 - \sqrt{1 - 4c}}{2}, p_+ = \frac{1 + \sqrt{1 - 4c}}{2}$
2. $I = [-p_+, p_+]$
3. $x_0 \not\in I \implies x_n \to \infty$
4. $\Lambda = \{x \in I \mid Q^n_c(x) \in I \text{ for all } n\}$
5. $A_1 = (-\sqrt{-c - p_+}, \sqrt{-c - p_+})$ is the open subinterval of $I$ that contains all the points $x_0$ such that $x_1 < -p_+; i.e. the orbit of $x_0$ under $Q_c$ ‘escapes to infinity’ after one iteration.
6. $A_1$ divides $I$ into two disjoint closed subintervals, $I_0$ and $I_1$: 

$$-p_+ \quad l_0 \quad A_1 \quad l_1 \quad p_+$$
Itineraries

Now let $x_0 \in \Lambda \subset l_0 \cup l_1$. For all $n$, $x_n = Q^n_c(x_0) \in l_0 \cup l_1$.

**Definition:** The itinerary of $x_0$ is the sequence $S(x_0)$ of 0’s and 1’s given by

$$S(x_0) = (s_0 s_1 s_2 s_3 \ldots s_n \ldots)$$

such that

$$s_n = \begin{cases} 0 & \text{if } x_n \in l_0 \\ 1 & \text{if } x_n \in l_1 \end{cases}.$$

The itinerary of $x_0$ is a simplified, symbolic representation of the orbit of $x_0$ under $Q_c$.

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**Examples**

**Example 1:** $S(p_+) = (1111\ldots)$ since $x_n = p_+ \in l_1$, for all $n$.

**Example 2:** $S(-p_+) = (0111\ldots)$ since $x_0 \in l_0$ but $x_n = p_+ \in l_1$, for all $n > 0$.

**Example 3:** $S(p_-) = (0000\ldots)$ since $x_n = p_- \in l_0$, for all $n$.

**Example 4:** If the first six terms in the orbit of $x_0$ under $Q_c$ look like

- $-p_+$
- $l_0$
- $-\sqrt{-c - p_+}$
- $A_1$
- $\sqrt{-c - p_+}$
- $l_1$
- $p_+$

then

$$S(x_0) = (001011\ldots)$$
What Is Sequence Space?

**Definition 1:** Sequence space is the set

\[ \Sigma = \{(s_0s_1s_2 \ldots s_n \ldots) \mid s_i = 0 \text{ or } 1, \text{ for } i = 0, 1, 2, \ldots, n, \ldots \}. \]

Every itinerary of every possible choice of \( x_0 \in \Lambda \) is in \( \Sigma \).

**Definition 2:** The distance between two points \( s = (s_0s_1s_2 \ldots) \) and \( t = (t_0t_1t_2 \ldots) \) in \( \Sigma \) is defined to be

\[ d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}. \]

This infinite series always converges since \( |s_i - t_i| = 0 \text{ or } 1 \); thus

\[ d[s, t] \leq \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - 1/2} = 2. \]

**Examples**

Let \( s = (000000 \ldots), t = (111111 \ldots), u = (010101 \ldots) \); then

1. \[ d[s, t] = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1 - 1/2} = 2. \]

2. \[ d[t, u] = 1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots = \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{1 - 1/4} = \frac{4}{3}. \]

3. \[ d[u, s] = \frac{1}{2} + \frac{1}{2^3} + \cdots = \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{4^i} = \frac{1}{2} \cdot \frac{1}{1 - 1/4} = \frac{2}{3}. \]
\( d \) is called a distance function, or metric, on \( \Sigma \)

\( d : \Sigma \rightarrow \mathbb{R} \) satisfies the following three conditions. For every \( s, t, u \in \Sigma \):

1. Non-negativity: \( d[s, t] \geq 0 \), and \( d[s, t] = 0 \iff s = t \).
2. Symmetry: \( d[s, t] = d[t, s] \)
3. Triangle Inequality: \( d[s, u] \leq d[s, t] + d[t, u] \)

\( \Sigma \) along with \( d \) is called a metric space. It is a space in which we can talk about distances, but it is not like the familiar spaces \( \mathbb{R} \) or \( \mathbb{R}^2 \). For instance, the farthest apart any two points can be in \( \Sigma \) is 2. This is not at all like \( \mathbb{R} \) in which you could choose two numbers on the real line as far apart as you like. Nevertheless, with \( d \) defined on \( \Sigma \) as above we can think geometrically in \( \Sigma \).

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Proof of the Three Properties

The proofs of the three properties of \( d \) depend on the analogous properties of absolute value, which is the function used in \( \mathbb{R} \) to define the distance between two real numbers \( a \) and \( b \), \( |a - b| \).

\[
d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \geq 0; \quad d[s, t] = 0 \iff |s_i - t_i| = 0 \iff s_i = t_i \iff s = t
\]

\[
d[s, t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=0}^{\infty} \frac{|t_i - s_i|}{2^i} = d[t, s]
\]

\[
d[s, u] = \sum_{i=0}^{\infty} \frac{|s_i - u_i|}{2^i} = \sum_{i=0}^{\infty} \frac{|s_i - t_i + t_i - u_i|}{2^i} \leq \sum_{i=0}^{\infty} \frac{|s_i - t_i| + |t_i - u_i|}{2^i}
\]

\[
= d[s, t] + d[t, u]
\]
The Proximity Theorem

With the above metric $d$ defined on $\Sigma$ we can determine when two sequences in sequence space are ‘close together’. In fact, it turns out to be easy to tell if two sequences are close together:

**The Proximity Theorem:** Let $s, t \in \Sigma$ such that $s_i = t_i$ for $0 \leq i \leq n$. Then

$$d[s, t] \leq \frac{1}{2^n}.$$ 

Conversely, if

$$d[s, t] < \frac{1}{2^n},$$

then $s_i = t_i$ for $0 \leq i \leq n$.

Ie. two sequences are close together if their first few entries agree.

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**Proof of the Proximity Theorem**

Suppose $s_i = t_i$ for $0 \leq i \leq n$, then

$$d[s, t] = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{2^{n+1}} \frac{1}{1 - 1/2} = \frac{1}{2^n}.$$ 

On the other hand, if $s_j \neq t_j$, for some $j \leq n$, then

$$d[s, t] \geq \frac{1}{2^j} \geq \frac{1}{2^n}.$$ 

Consequently, if

$$d[s, t] < \frac{1}{2^n},$$

then $s_i = t_i$ for all $0 \leq i \leq n$. 
Calculus Review: Limits and Continuity

Recall, from MAT137H1Y, the following definitions:

1. \( \lim_{x \to a} f(x) = L \) if and only if for all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that \( 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon \).

2. \( f \) is continuous at the point \( c \) if \( \lim_{x \to c} f(x) = f(c) \).

3. \( f \) is continuous on the open interval \((a, b)\) if \( \lim_{x \to c} f(x) = f(c) \) for all \( c \in (a, b) \).

4. \( f \) is continuous on the closed interval \([a, b]\) if \( \lim_{x \to c} f(x) = f(c) \) for all \( c \in (a, b) \), and
   \[ \lim_{x \to a^+} f(x) = f(a) \] and \( \lim_{x \to b^-} f(x) = f(b) \).

5. \( f : \mathbb{R} \to \mathbb{R} \) is continuous on \( \mathbb{R} \) if it is continuous at every point \( c \in \mathbb{R} \).

Four Definitions, and New Ways to Describe Continuity

**Definition 1:** A subset \( A \subset \mathbb{R} \) is open if it is a union of open intervals.

**Definition 2:** A subset \( A \subset \mathbb{R} \) is closed if it is the complement of an open set. That is, \( \mathbb{R} - A \) is open.

**Definition 3:** The closure of a subset \( A \subset \mathbb{R} \) is \( \bar{A} \), the intersection of all closed sets that contain \( A \).

**Definition 4:** If \( Y \) is a subset of \( \mathbb{R} \) and \( f : \mathbb{R} \to \mathbb{R} \), then
\[ f^{-1}(Y) = \{ x \in \mathbb{R} \mid f(x) \in Y \} \]

**Theorem:** Suppose \( f : \mathbb{R} \to \mathbb{R} \) is a function. The following four statements are equivalent:

1. \( f \) is continuous on \( \mathbb{R} \).

2. \( f^{-1}(A) \) is an open set for any open set \( A \subset \mathbb{R} \).

3. For every subset \( A \subset \mathbb{R} \), \( f(\bar{A}) \subset \bar{f(A)} \).

4. \( f^{-1}(A) \) is a closed set for any closed set \( A \subset \mathbb{R} \).
Examples and Comments

1. $\mathbb{R}$ and $\phi$ are both considered open and closed.
2. The union of any number of open sets is an open set; the intersection of any finite number of open sets is open.
3. The intersection of any number of closed sets is a closed set; the union of any finite number of closed sets is closed.
4. $(a, b) = [a, b]$; if $A = \{1/n \mid n \in \mathbb{N}\}$, then $\bar{A} = A \cup \{0\}$.
5. Suppose $f$ is continuous at $x = c$, and use Property 2 of the above Theorem. For all $\epsilon > 0$, $A = (f(c) - \epsilon, f(c) + \epsilon)$ is an open interval, so $f^{-1}(A)$ is an open set. Since $c \in f^{-1}(A)$, there is an open interval $(a, b)$ such that $c \in (a, b)$. Pick $\delta = \min \{c - a, b - c\}$. Thus continuity of $f$ at $x = c$ implies

$$\lim_{x \to c} f(x) = f(c).$$

Open Intervals and Open Sets in $\Sigma$

Using the distance function $d$, as defined in Section 9.2, we can define open sets in $\Sigma$.

1. An open interval of radius $\epsilon$ centered at $a \in \Sigma$ will be the set

$$\{s \in \Sigma \mid d[s, a] < \epsilon\}.$$

2. An open set in $\Sigma$ is the union of any number of open intervals in $\Sigma$.

Note that if $\epsilon = 2^{-n}$, then $s$ is in the open interval of radius $\epsilon$ centered at $a$ if

$$s_i = a_i, \text{ for } 0 \leq i \leq n,$$

by the Proximity Theorem. That is, the first $n + 1$ entries of $s$ must be the same as the first $n + 1$ entries of $a$. 
Definition of the Shift Map

**Definition:** The shift map \( \sigma : \Sigma \rightarrow \Sigma \) is defined by

\[
\sigma(s_0s_1s_2s_3s_4 \ldots) = (s_1s_2s_3s_4 \ldots).
\]

That is, for \( s \in \Sigma \), \( \sigma(s) \) is obtained from \( s \) by dropping its first entry. Thus:

1. \( \sigma(010101 \ldots) = (10101 \ldots) \)
2. \( \sigma(011111 \ldots) = (11111 \ldots) \)
3. \( \sigma(001011 \ldots) = (01011 \ldots) \)

Iterating the Shift Map

It is easy to iterate the shift map: we just keep dropping the first entry at each step. For example:

1. \( \sigma^2(s_0s_1s_2s_3s_4 \ldots) = \sigma(s_1s_2s_3s_4 \ldots) = (s_2s_3s_4s_5 \ldots) \)
2. \( \sigma^3(s_0s_1s_2s_3s_4 \ldots) = \sigma(s_2s_3s_4s_5 \ldots) = (s_3s_4s_5s_6 \ldots) \)
3. \( \sigma^n(s_0s_1s_2s_3s_4 \ldots) = (s_ns_{n+1}s_{n+2}s_{n+3} \ldots) \)

Notation: if

\[
s = (s_0s_1 \ldots s_{n-1}s_0s_1 \ldots s_{n-1}s_0s_1 \ldots s_n \ldots)
\]

is a repeating sequence, we shall write:

\[
s = (s_0 \ldots s_{n-1}).
\]
The Periodic Points of the Shift Map

If \( s = (s_0 s_1 \ldots s_{n-1}) \) is a repeating sequence then

\[
\sigma^n(s) = s.
\]

Conversely, any periodic point of period \( n \) for \( \sigma \) must be a repeating sequence. This is so much easier than it is for \( Q_c \). We can actually write down all the periodic points for \( \sigma \):

1. the only two fixed points for \( \sigma \) are \((11111\ldots)\) or \((00000\ldots)\)
2. the only two points for \( \sigma \) of prime period 2 are \((01)\) or \((10)\), and they form a 2-cycle: \( \sigma(01) = (10) \), \( \sigma(10) = (01) \)
3. there are only two 3-cycles for \( \sigma \):

\[
(001) \rightarrow (010) \rightarrow (100) \rightarrow (001);
\]

and

\[
(110) \rightarrow (101) \rightarrow (011) \rightarrow (110)
\]

Continuity of the Shift Map

\( \sigma : \Sigma \longrightarrow \Sigma \) is continuous in the sense that for any \( s \in \Sigma \), and for all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that

\[
d[s, t] < \delta \Rightarrow d[\sigma(s), \sigma(t)] < \epsilon.
\]

Proof: pick \( n \) large enough so that \( 1/2^n < \epsilon \); pick \( \delta = 1/2^{n+1} \). Then, by the Proximity Theorem,

\[
d[s, t] < 1/2^{n+1} \Rightarrow t_i = s_i, 0 \leq i \leq n + 1
\]

\[
\Rightarrow \sigma(t) = (s_1 s_2 s_3 \ldots s_{n+1} t_{n+2} t_{n+3} \ldots)
\]

\[
\Rightarrow d[\sigma(s), \sigma(t)] \leq \frac{1}{2^n} < \epsilon,
\]

making use of the Proximity Theorem, again.
Connections, Connections ... and a Commuting Diagram

We have $Q_c : \Lambda \rightarrow \Lambda, \quad \sigma : \Sigma \rightarrow \Sigma$ and $S : \Lambda \rightarrow \Sigma$. What are the connections between these three functions?

**Theorem:** If $x \in \Lambda$, then

\[(S \circ Q_c)(x) = (\sigma \circ S)(x),\]

i.e. $S \circ Q_c = \sigma \circ S$ and the following diagram commutes:

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{Q_c} & \Lambda \\
\downarrow S & & \downarrow S \\
\Sigma & \xrightarrow{\sigma} & \Sigma
\end{array}
\]

**Proof:** Let $x \in \Lambda$ and suppose $S(x) = (s_0s_1s_2s_3\ldots)$. This means

\[Q^n_c(x) \in I_{s_n}, \text{ for } n \geq 0,
\]

where $I_{s_n}$ is either $I_0$ or $I_1$, depending on $s_n$. That is,

\[Q_c(x) \in I_{s_1}, Q^2_c(x) \in I_{s_2}, Q^3_c(x) \in I_{s_3}, \ldots
\]

and the itinerary of $Q_c(x)$ is $(s_1s_2s_3\ldots)$. Consequently

\[S(Q_c(x)) = (s_1s_2s_3s_4\ldots) = \sigma(s_0s_1s_2s_3\ldots) = \sigma(S(x)),
\]

or equivalently

\[(S \circ Q_c)(x) = (\sigma \circ S)(x).
\]
More Commuting Diagrams

whence $S \circ Q^2_c = \sigma^2 \circ S$.

whence $S \circ Q^3_c = \sigma^3 \circ S$. In general $S \circ Q^n_c = \sigma^n \circ S$.

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Orbits of $x$ under $Q_c$ and Orbits of $S(x)$ under $\sigma$

So $S$ converts orbits of $x$ under $Q_c$ to orbits of $S(x)$ under $\sigma$. That is, $S$ takes the orbit of $x$ under $Q_c$, which is hard to calculate:

$$x, Q_c(x), Q^2_c(x), Q^3_c(x), \ldots, Q^n_c(x), \ldots$$

to

$$S(x), \sigma(S(x)), \sigma^2(S(x)), \sigma^3(S(x)), \ldots, \sigma^n(S(x)), \ldots,$$

which is easy to calculate, assuming you know what $S(x)$ is!

Actually, we hope that by knowing things about orbits under $\sigma$ we will be able to ‘go the other way’ and say something about the orbits under $Q_c$. But this requires some further properties of $S$. 

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Further Properties of \( S : \Lambda \rightarrow \Sigma \)

The following properties of \( S \) can be proved:

1. \( S \) is one-to-one: \( S(x) = S(y) \Rightarrow x = y \)
2. \( S \) is onto: for each \( s \in \Sigma \) there is \( x \in \Lambda \) such that \( S(x) = s \)
3. \( S \) is continuous in the following sense: If \( x \in \Lambda \), then for all \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[
    x, y \in \Lambda, |y - x| < \delta \Rightarrow d[S(y), S(x)] < \epsilon.
\]

4. \( S^{-1} : \Sigma \rightarrow \Lambda \) is also continuous.

\( S : \Lambda \rightarrow \Sigma \) is One-to-One

Suppose \( x \neq y \) but \( S(x) = S(y) \); then \( Q_c^n(x) \) and \( Q_c^n(y) \) are both in the same interval, \( I_0 \) or \( I_1 \), for each value of \( n \). As we saw in Section 7.2, if

\[
    c \leq -\frac{5 + 2\sqrt{5}}{4},
\]

then there is a number \( \mu > 1 \) such that the length of the interval \( H = [Q_c^n(x), Q_c^n(y)] \) (or \( [Q_c^n(y), Q_c^n(x)] \)) is greater than

\[
    \mu^n |x - y|.
\]

If \( n \) is large enough this length is greater than the length of \( I \), contradicting the fact the interval \( H \) must be contained within \( I \). Note: as in Section 7.2, this result is actually true for all \( c < -2 \).
$S : \Lambda \longrightarrow \Sigma$ is Onto; Use $F^{-1}(A \cap B) = F^{-1}(A) \cap F^{-1}(B)$

Let $s \in \Sigma$; we shall construct $x \in \Lambda$ such that $S(x) = s$. Let

$$
I_{s_0s_1...s_n} = \{x \in I \mid x \in I_{s_0}, Q_c(x) \in I_{s_1}, \ldots, Q_c^n(x) \in I_{s_n}\}
$$

$$
= I_{s_0} \cap Q_c^{-1}(I_{s_1}) \cap \cdots \cap Q_c^{-n}(I_{s_n})
$$

$$
= I_{s_0} \cap Q_c^{-1}(I_{s_1} \cap \cdots \cap Q_c^{-n}(I_{s_n}))
$$

By induction, on the number of subscripts in $s_0s_1...s_n$, you can prove $I_{s_0s_1...s_n}$ is always a single closed interval. Moreover, these closed subintervals are nested because

$$
I_{s_0s_1...s_n} = I_{s_0s_1...s_{n-1}} \cap Q_c^{-n}(I_{s_n}) \subset I_{s_0s_1...s_{n-1}}
$$

Thus $\bigcap_{n \geq 0} I_{s_0s_1...s_n} \neq \emptyset$; and $x \in \bigcap_{n \geq 0} I_{s_0s_1...s_n} \Rightarrow x \in \Lambda, S(x) = s$.

$S : \Lambda \longrightarrow \Sigma$ is Continuous

Let $\epsilon > 0$ be given and pick $n$ such that $1/2^n < \epsilon$.

Let $J_n$ be a closed interval such that $J_n \subset I_{s_n}$, $Q_c^n(x) \in J_n$. Then $Q_c^{-1}(J_n)$ consists of two closed intervals (why?), one in $I_0$ and one in $I_1$; let $J_{n-1}$ be the closed interval in $I_{s_{n-1}}$ that contains $Q_c^{-1}(x)$. Proceed in this fashion for $0 \leq i \leq n$ to obtain closed intervals $J_i \subset I_{s_i}$, such that $Q_c^i(x) \in J_i$. Then

$$
x, y \in J_0 \Rightarrow Q(x), Q(y) \in J_1 \Rightarrow \cdots \Rightarrow Q_c^n(x), Q_c^n(y) \in J_n.
$$

Consequently, $x, y \in \Lambda \cap J_0 \Rightarrow d[S(x), S(y)] \leq 1/2^n < \epsilon$. 

Since $S : \Lambda \rightarrow \Sigma$ is one-to-one and onto, $S^{-1} : \Sigma \rightarrow \Lambda$ exists.

Hence $Q_c \circ S^{-1} = S^{-1} \circ \sigma$. In general,

$$Q^n_c \circ S^{-1} = S^{-1} \circ \sigma^n.$$ 

Thus there is a one-to-one correspondence between the orbits of $x \in \Lambda$ under $Q_c$ and the orbits of $s \in \Sigma$ under $\sigma$.

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$S^{-1}$ is continuous, in the sense that for any $s \in \Sigma$, and all $\epsilon > 0$, there is a $\delta > 0$, such that $d[s, t] < \delta \Rightarrow |S^{-1}(s) - S^{-1}(t)| < \epsilon$.

**Proof:** let $s = S(x)$, $t = S(y)$. Then $d[s, t] < 1/2^n$ means $Q_i^c(x)$ and $Q_i^c(y)$ are always in the same interval $I_0$ or $I_1$ for $0 \leq i \leq n$. As we saw in Section 7.2, for $c < -2.368$, there is $\mu > 1$ such that

$$|Q^n_c(x) - Q^n_c(y)| \geq \mu^n |x - y| \iff |x - y| \leq \frac{|Q^n_c(x) - Q^n_c(y)|}{\mu^n}$$

But $|Q^n_c(x) - Q^n_c(y)| < L$, where $L$ is the length of $I_1$. (Why?) Pick $n$ large enough so that $L/\mu^n < \epsilon$. Then

$$d[S(x), S(y)] < 1/2^n \Rightarrow |x - y| < \epsilon \iff |S^{-1}(S(x)) - S^{-1}(S(y))| < \epsilon.$$
What Is a Homeomorphism?

Suppose \( h : X \rightarrow Y \) and \( d_1 \) and \( d_2 \) are distance functions defined on \( X \) and \( Y \), respectively. Then \( h \) is called a homeomorphism if

1. \( h \) is one-to-one: \( h(x) = h(y) \Rightarrow x = y \)
2. \( h \) is onto: if \( y \in Y \) there is an \( x \in X \) such that \( h(x) = y \)
3. \( h \) is continuous on \( X \): for all \( x \in X \) and all \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( d_1[x, y] < \delta \Rightarrow d_2[h(x), h(y)] < \epsilon \)
4. \( h^{-1} : Y \rightarrow X \) is also continuous.

Equivalently: \( h : X \rightarrow Y \) is a homeomorphism if it is a bijection that maps open sets to open sets, and closed sets to closed sets. Consequently: \( x_1 \) and \( x_2 \) are close together in \( X \) if and only if \( h(x_1) \) and \( h(x_2) \) are close together in \( Y \).

Examples: \( S : \Lambda \rightarrow \Sigma \) is a homeomorphism. More mundanely, \( h(x) = x^3 \), \( h : \mathbb{R} \rightarrow \mathbb{R} \), is a homeomorphism.

Conjugacy

**Definition:** Let \( F : X \rightarrow X \) and \( G : Y \rightarrow Y \) be two functions. \( F \) and \( G \) are called conjugate if there is a homeomorphism \( h : X \rightarrow Y \) such that \( h \circ F = G \circ h \); \( h \) is called a conjugacy.

\[
\begin{array}{ccc}
X & \xrightarrow{F} & X \\
\downarrow h & \quad & \downarrow h \\
Y & \xrightarrow{G} & Y \\
\end{array}
\]

\[ h \circ F = G \circ h. \]

\[
F^n = h^{-1} \circ G^n \circ h \\
G^n = h \circ F^n \circ h^{-1}
\]
The Conjugacy Theorem

We have proved

**Theorem:** The shift map \( \sigma \) on \( \Sigma \) is conjugate to \( Q_c \) on \( \Lambda \); the conjugacy is \( S \).

\[
\begin{array}{ccc}
\Lambda & \xrightarrow{Q_c} & \Lambda \\
S & \searrow & S \\
\Sigma & \xrightarrow{\sigma} & \Sigma
\end{array}
\]

\[S \circ Q_c = \sigma \circ S\]

\[Q^n_c = S^{-1} \circ \sigma^n \circ S\]

\[\sigma^n = S \circ Q^n_c \circ S^{-1}\]

This means that dynamics of \( \sigma \) on \( \Sigma \) and \( Q_c \) on \( \Lambda \) are essentially the same. For example:

1. \( S \) converts orbits of \( x \) under \( Q_c \) to orbits of \( S(x) \) under \( \sigma \)
2. \( S^{-1} \) converts orbits of \( s \) under \( \sigma \) to orbits of \( S^{-1}(s) \) under \( Q_c \)
3. If \( s \) is a periodic point for \( \sigma \) then \( S^{-1}(s) \) is a periodic point for \( Q_c \), with the same period.
4. If \( s \) is an eventually periodic point for \( \sigma \) then \( S^{-1}(s) \) is an eventually periodic point for \( Q_c \)

Eg. \( \sigma(s) = s \Rightarrow S^{-1}(\sigma(s)) = S^{-1}(s) \Rightarrow Q_c(S^{-1}(s)) = S^{-1}(s) \).
How Can We Define Chaos?

If something is truly chaotic then it should defy all definition or description. A chaotic dynamical system appears chaotic in the colloquial sense, but actually satisfies some definite criteria in the mathematical sense. Those three criteria are

1. density
2. transitivity
3. sensitivity

First we shall define these three notions, and then we shall give some examples of dynamical systems that satisfy these three criteria.

What Is a Dense Set?

**Definition:** Suppose $Y$ is a subset of $X$, which has distance function $d$. $Y$ is said to be dense in $X$ if for every open interval $A \subset X$,

$$Y \cap A \neq \emptyset.$$

Equivalently:

1. if $x \in X$ then there is a sequence $y_n \in Y$ such that

$$\lim_{n \to \infty} y_n = x.$$

2. $\overline{Y} = X$; i.e. the closure of $Y$ is $X$.

What is open, or closed, and the calculation of limits, all depend on the distance function $d$. Intuitively, $Y$ is a dense subset of $X$ if for any point $x \in X$ there is a point $y \in Y$ arbitrarily close to $x$. 
Examples of Dense Sets in $\mathbb{R}$

For $\mathbb{R}$ the usual distance function is absolute value: the distance between $a, b \in \mathbb{R}$ is $|a - b|$; and a typical open interval in $\mathbb{R}$ is $(a, b)$. Examples:

1. The rational numbers $\mathbb{Q}$ are dense in the real numbers $\mathbb{R}$.
   **Proof:** suppose $x = a_j \ldots a_1 a_0 \cdot b_1 b_2 \ldots b_n \ldots$ is an irrational number. Then the sequence of $y_n = a_j \ldots a_1 a_0 \cdot b_1 b_2 \ldots b_n$ is a sequence of rational numbers with limit $x$. (Think of $\sqrt{2} = 1.4142135623730950488 \ldots$; take $y_1 = 1.4, y_2 = 1.41, y_3 = 1.414$, and so on.)

2. $(a, b)$ is dense in $[a, b]$

3. The integers $\mathbb{Z}$ are not dense in $\mathbb{R}$ or $\mathbb{Q}$; no sequence of integers can have limit $1/2$, for instance.

Examples of Dense Sets in $\Sigma$

**Example 1:** The set of periodic points in $\Sigma$ is dense in $\Sigma$: Let $s = (s_0 s_1 \ldots s_n s_{n+1} \ldots)$ be an arbitrary sequence in $\Sigma$, let $\varepsilon > 0$. Pick $n$ such that $1/2^n < \varepsilon$; let $t_n = (s_0 s_1 \ldots s_n)$. Then $t_n$ is a periodic point in $\Sigma$ and, by the Proximity Theorem,

$$d[s, t_n] \leq 1/2^n < \varepsilon.$$

**Example 2:** The orbit of

$$\hat{s} = (0 10 00 01 101 110 000 010 101 \ldots \ldots)$$

under $\sigma$ is dense in $\Sigma$: Given $s \in \Sigma$ pick $k$ such that the $n + 1$ block at the beginning of $\sigma^k(\hat{s})$ is the same as $(s_0 s_1 \ldots s_n)$. Now apply the Proximity Theorem.
What Is Transitivity?

**Definition:** A dynamical system \( F : X \rightarrow X \) is transitive if for any points \( x, y \in X \) and any \( \epsilon > 0 \), there is a third point \( z \), within \( \epsilon \) of \( x \), whose orbit comes within \( \epsilon \) of \( y \). That is, for any \( x, y \in X \) there is a \( z \in X \) and an \( n > 0 \) such that

\[
    d[z, x] < \epsilon \text{ and } d[F^n(z), y] < \epsilon.
\]

In other words, a transitive dynamical system has the property that there is always an orbit that gets arbitrarily close to any two given points.

**Example**

\( \sigma : \Sigma \rightarrow \Sigma \) is transitive, because as we have seen, the orbit of

\[
    \hat{s} = (0, 1, 00011011, 000010101\ldots\ldots)
\]

under \( \sigma \) is dense in \( \Sigma \). So, if \( s, t \in \Sigma \), and \( \epsilon > 0 \),

1. then there is a \( k \) such that \( d[s, \sigma^k(\hat{s})] < \epsilon \)
2. and some \( n \) steps later, \( d[t, \sigma^{k+n}(\hat{s})] < \epsilon \)

In fact, any dynamical system that has a dense orbit must be transitive.
What Is Sensitivity to Initial Conditions?

**Definition:** A dynamical system $F : X \rightarrow X$ depends sensitively on initial conditions if there is a $\beta > 0$ such that for every $x \in X$ and any $\epsilon > 0$ there is a $y \in X$ within $\epsilon$ of $x$ and a $k$ such that the distance between $F^k(x)$ and $F^k(y)$ is at least $\beta$. That is, for every $x \in X$ and any $\epsilon > 0$ there is a $y \in X$ and a $k$ such that

$$d[y, x] < \epsilon \text{ but } d[F^k(y), F^k(x)] \geq \beta.$$ 

We can find $y$ as close to $x$ as we like but eventually the orbit of $x$ and $y$ under $F$ will be separated by more than $\beta$. Putting it yet another way: there is no guarantee that choosing $x$ and $y$ close together will ensure the orbits of $x$ and $y$ under $F$ stay close together. In terms of numerical calculations: a small round-off error can result in a totally different orbit being calculated.

Example of a Sensitive Dynamical System

We shall see shortly that

$$Q_{-2} : [-2, 2] \rightarrow [-2, 2]$$

is a sensitive dynamical system, with of course

$$Q_{-2}(x) = x^2 - 2.$$ 

The diagram to the right illustrates the first 30 iterations for the three orbits of $x = 0.09, 0.1, 0.11$. 
Example of an Insensitive Dynamical System

\[ C : \mathbb{R} \rightarrow \mathbb{R} \text{ defined by} \]

\[ C(x) = \cos x \]

possesses no sensitivity to initial conditions whatsoever. As we have seen, for any choice of \( x \) the orbit of \( x \) under \( C \) converges to the unique fixed point of \( C \), approximately 0.73908. Orbits that start together stay together. The diagram to the right illustrates three orbits for \( x = -0.9, -0.8, -0.7 \).

**Definition of A Chaotic Dynamical System**

**Definition:** A dynamical system \( F : X \rightarrow X \) is chaotic if

1. The set of all periodic points for \( F \) is dense in \( X \).
2. \( F \) is transitive.
3. \( F \) depends sensitively on initial conditions.

**Example:** The shift map \( \sigma : \Sigma \rightarrow \Sigma \) is chaotic. We have already proved properties 1 and 2; we only need to show that \( \sigma \) depends sensitively on initial conditions.
$\sigma : \Sigma \longrightarrow \Sigma$ Depends Sensitive On Initial Conditions

Let $\beta = 1$; for any $s \in \Sigma$, and any $\epsilon > 0$, pick $n$ large enough so that $1/2^n < \epsilon$. Suppose $t \in \Sigma$ and $d[t, s] < 1/2^n$. If $t \neq s$, then there is a $k > n$ such that $t_k \neq s_k$. Thus $|t_k - s_k| = 1$ and consequently

$$d[\sigma^k(t), \sigma^k(s)] = \sum_{j=0}^{\infty} \frac{|t_k+j - s_k+j|}{2^j} \geq |t_k - s_k| = 1.$$  

This actually proves more than sensitivity to initial conditions of the shift map; it proves that all other points $t \neq s$ have orbits that eventually separate by at least 1 from the orbit of $s$ under $\sigma$.

---

**The Density Proposition**

Before we can prove that $Q_c$ is chaotic on $\Lambda$ we need one more preliminary result.

**Theorem:** Suppose $F : X \longrightarrow Y$ is a continuous map that is onto and suppose that $D$ is a dense subset of $X$. Then $F(D)$ is a dense subset of $Y$.

**Proof:** Let $B$ be an open set in $Y$. Since $F$ is onto and continuous, $A = F^{-1}(B)$ is a non-empty open set in $X$; so $A \cap D \neq \phi$. Pick $x \in A \cap D$. Then $F(x) \in F(A) \cap F(D) = B \cap F(D)$. Hence $B \cap F(D) \neq \phi$. 

---
\[ Q_c : \Lambda \longrightarrow \Lambda \text{ is a chaotic dynamical system, if } c < -2 \]

\[ \Lambda \xrightarrow{Q_c} \Lambda \]

\[ S^{-1} : \Sigma \longrightarrow \Lambda \text{ is also a homeomorphism, and} \]

\[ S^{-1}(s) \text{ is a periodic point in } \Lambda \text{ if and only if } s \text{ is a periodic point in } \Sigma. \]

Then, by the Density Proposition, \( S^{-1} \) maps the dense set of periodic points for \( \sigma \) in \( \Sigma \) to a dense set of periodic points for \( Q_c \) in \( \Lambda \).

Secondly, since the orbit of \( \hat{s} \) under \( \sigma \) is dense in \( \Sigma \), the Density Proposition ensures that the orbit of \( S^{-1}(\hat{s}) \) under \( Q_c \) is also dense in \( \Lambda \). This means \( Q_c \) is also transitive.

It only remains to show that \( Q_c \) is sensitive to initial conditions.

\[ \Lambda \xrightarrow{Q_c} \Lambda \]

Recall that \( l_0 \) and \( l_1 \) are the closed disjoint subintervals of \( I = [-p_+, p+] \), produced by discarding all the points in the open interval \( A_1 \). Let \( \beta \) be the length of the interval \( A_1 \).

Now let \( x, y \in \Lambda, x \neq y \).

Since \( S \) is bijective \( S(x) \neq S(y) \), and there is a \( k \) such that the \( k \)-th entries of \( S(x) \) and \( S(y) \) differ. This means that both \( F^k(x) \) and \( F^k(y) \) are not in the same interval, \( l_0 \) or \( l_1 \). Consequently,

\[ |F^k(x) - F^k(y)| \geq \beta. \]

Therefore the orbit of \( y \) under \( Q_c \) for any \( y \neq x \) eventually separates from the orbit of \( x \) under \( Q_c \) by at least \( \beta \).
Another Chaotic Dynamical System

Let

\[ V(x) = 2|x| - 2. \]

Check that

\[ V : [-2, 2] \longrightarrow [-2, 2]; \]

or look at the graph of \( V \) to the right.

We claim that \( V : [-2, 2] \longrightarrow [-2, 2] \) is a chaotic dynamical system. To see why we need only consider graphs of \( V^n \).

Higher Iterations of \( V \)

In general, the graph of \( V^n \) maps \([-2, 2]\) to itself, consists of \( 2^n \) line segments with slope \( \pm 2^n \), each of which maps an interval of length \( 1/2^{n-2} \) onto the interval \([-2, 2]\).
Periodic Points of $V$ are Dense in $[-2, 2]$

Let $J$ be an arbitrary open interval in $[-2, 2]$. Pick a closed interval $J_n$ of length $1/2^{n-2}$ such that $J_n \subset J$ and $J_n$ is one of the subintervals in $[-2, 2]$ on which

$$V^n : J_n \rightarrow [-2, 2].$$

Since every segment of $V^n$ intersects the line $y = x$, there is an $x \in J_n$ such that

$$V^n(x) = x.$$

Since $J_n \subset J$, $J$ contains a periodic point of $V$.

$V : [-2, 2] \rightarrow [-2, 2]$ Is Transitive

Pick any two points $x, y \in [-2, 2]$, and let $\epsilon > 0$ be given. Pick $n$ large enough so that $1/2^{n-2} < \epsilon$. As in the previous slide, pick a closed interval $J_n$ of length $1/2^{n-2}$ such that $x \in J_n$ and

$$V^n : J_n \rightarrow [-2, 2].$$

Since $y \in [-2, 2]$, there is a $z \in J_n$ such that $V^n(z) = y$. Then

$$|x - z| < 1/2^{n-2} < \epsilon$$

and

$$|y - V^n(z)| = 0 < \epsilon.$$

So not only does the orbit of $z$ under $V$ get close to $y$, it actually hits $y$. In any event, $V$ is transitive on $[-2, 2]$. 
**V** Depends Sensitive on Initial Conditions

Take $\beta = 2$, let $\epsilon > 0$ and suppose $x \in [-2, 2]$. Pick $n$ large enough so that $1/2^{n-2} < \epsilon$. As in the previous two slides, pick a closed interval $J_n$ of length $1/2^{n-2}$ such that $x \in J_n$ and

$$V^n : J_n \rightarrow [-2, 2].$$

Pick $y \in J_n$ such that

$$|y - x| \geq \frac{1}{2} \text{length}(J_n) = \frac{1}{2^{n-1}},$$

which is possible because $x$ lies in one half of $J_n$. Now apply MVT to $V^n$ on the interval $[x, y]$ (or $[y, x]$): there is $c \in [x, y]$ such that

$$|V^n(y) - V^n(x)| = |(V^n)'(c)||y - x| \geq \frac{2^n}{2^{n-1}} = 2.$$  

---

Is $Q_{-2} : [-2, 2] \rightarrow [-2, 2]$ A Chaotic Dynamical System?

Let $C(x) = -2\cos(\pi x/2)$. Then $Q_{-2} \circ C = C \circ V$:

$$[\begin{array}{c}
[-2, 2] \\
c
\end{array} \xrightarrow{V} \begin{array}{c}
[-2, 2] \\
c
\end{array} \xrightarrow{C} \begin{array}{c}
[-2, 2] \\
Q_{-2}
\end{array}$$

$$C(V(x)) = C(2|x| - 2) = -2\cos(\pi |x| - \pi) = 2\cos(\pi x),$$

making use of some trig, and

$$Q_{-2}(C(x)) = Q_{-2}(-2\cos(\pi x/2)) = 4\cos^2(\pi x/2) - 2 = 2\cos(\pi x).$$

It seems as if $C$ is a conjugacy, whence $Q_{-2}$ is chaotic, since $V$ is. Actually $C$ is not one-to-one, but it is continuous and onto, so the Density Proposition still applies to $C$. Although $C$ is two-to-one, it still takes periodic points of $V$ to periodic points of $Q_{-2}$. So $C$ can be used to prove that $Q_{-2}$ is chaotic. $C$ is called a semiconjugacy.
Semiconjucagy

Suppose $F : X \rightarrow X$ and $G : Y \rightarrow Y$ are two dynamical systems. A mapping $h : X \rightarrow Y$ is called a semiconjucacy if $h$ is continuous, onto, at most $n$-to-one, and satisfies $h \circ F = G \circ h$.

Then $h(F^k(x)) = G^k(h(x))$. Thus a semiconjucacy takes orbits of $x$ under $F$ to orbits of $h(x)$ under $G$; and takes cycles of $F$ to cycles of $G$, although its prime period may become less.

Since $h$ is continuous and onto, the Density Proposition ensures the periodic points of $G$ are dense in $Y$, if the periodic points of $F$ are dense in $X$. Similarly, if $F$ has a dense orbit in $X$, then $G$ has a dense orbit in $Y$, although it may have many less distinct points. Semiconjucacies do usefully relate one system to another.

Another Doubling Function, Reference Page 125

Devaney calls it $D$, but I'm going to call it $D_2$, because it is just the square function \ldots applied to a complex variable $z$. That is:

1. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$
2. Define $D_2 : S^1 \rightarrow S^1$ by
   
   $D_2(z) = z^2$.
3. The orbit of $z = i$ under $D_2$ is eventually fixed: $i, -1, 1, 1, \ldots$
4. The orbit of $z = (1 + \sqrt{3}i)/2$ under $D_2$ is eventually a 2-cycle, as shown on the figure.

Figure: Unit Circle in $\mathbb{C}$
Why Does Devaney Call $D_2$ a Doubling Function?

If $z = \cos \theta + i \sin \theta \in S^1$, then $z^2 = \cos(2\theta) + i \sin(2\theta)$. So the argument $\theta$ of $z$ is doubling. Now consider $B : S^1 \rightarrow [-2, 2]$ defined by $B(z) = 2\Re(z)$. Check that $B \circ D_2 = Q^-_2 \circ B$:

\[
\begin{align*}
Q^-_2(B(z)) &= Q^-_2(2\cos \theta) \\
&= 4\cos^2 \theta - 2 \\
&= 2\cos(2\theta) \\
&= B(z^2)
\end{align*}
\]

Since $B$ is two-to-one, $B$ is a semiconjugacy; we can use it to show $D_2$ is chaotic on $S^1$ since $Q^-_2$ is chaotic on $[-2, 2]$. But it's tricky since we have to use $B^{-1}$, not a function, to get information about $D_2$ from $Q^-_2$. (See Chapter 16 for a more direct approach.)

Using Some Properties of Complex Numbers

Note: $z$ is a periodic point for $D_2$ iff $z$ is; and $B(z) = B(\bar{z})$.  

**Density**: suppose the periodic points of $D_2$ are not dense in $S^1$. Then there is an open interval $U \subset S^1$ that contains no periodic point for $D_2$. But $B(U)$ is an open interval in $[-2, 2]$ and so must contain a periodic point for $Q^-_2$, say $x$. Let $B^{-1}(x) = \{z, \bar{z}\}$. Then

\[
B(D^n_2(z)) = Q^n_-2(B(z)) = Q^n_-2(x) = x \Rightarrow D^n_2(z) = z \text{ or } \bar{z}.
\]

If $D^n_2(z) = z$ then $z$ is a periodic point; if $D^n_2(z) = \bar{z}$, then $D^{2n}_2(z) = z$; either way $z$ and $\bar{z}$ are periodic points for $D_2$. By continuity, $B^{-1}(B(U))$ is an open set, $U \cup \overline{U}$, and one of $z$ or $\bar{z}$ is in $U$, contradicting our assumption that $U$ contains no periodic points. Proving **Transitivity** and **Sensitivity** is left to the reader.