Chapter 3: Orbits
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**Functional Iteration**

If $F : \mathbb{R} \rightarrow \mathbb{R}$, then we shall write

- $F^2(x) = (F \circ F)(x) = F(F(x))$
- $F^3(x) = (F \circ F^2)(x) = F(F^2(x)) = F(F(F(x)))$
- $F^n(x) = (F \circ F^{n-1})(x) = F(F^{n-1}(x))$, for $n \geq 3$

We could in fact include the cases $n = 0$ and $n = 1$:

$F^0(x) = x$ and $F^1(x) = F(x)$.

$F^n(x)$ is called the $n$-th iterate of $F$, for $n \geq 0$.

Note: in this course $F^n(x)$ does not represent multiplication of functions, as opposed to what you may already be used to with $\sin^2(x) = \sin x \sin x = (\sin x)^2$. For us, $\sin^2(x) = \sin(\sin x)$.

**Examples**

1. Let $F(x) = x^2$.
   Then $F^2(x) = F(F(x)) = F(x^2) = (x^2)^2 = x^4$.
   $F^3(x) = F(F^2(x)) = F(x^4) = (x^4)^2 = x^8$. In this case it is easy to generalize:
   $$F^n(x) = x^{2^n}.$$  

2. Let $F(x) = x^2 + 1$.
   Then $F^2(x) = F(F(x)) = F(x^2 + 1) = (x^2 + 1)^2 + 1$.
   Expanding gives $F^2(x) = x^4 + 2x^2 + 2$.
   $F^3(x) = F(F^2(x)) = F(x^4 + 2x^2 + 2) = (x^4 + 2x^2 + 2)^2 + 1$.
   Expanding gives $F^3(x) = x^8 + 4x^6 + 8x^4 + 8x^2 + 5$. In this case it is not so easy to generalize.
Orbits and Dynamical Systems

Let \( x_0 \in \mathbb{R} \). We define the orbit of \( x_0 \) under \( F \) to be the sequence of points

\[ x_0, x_1, x_2, x_3, \ldots, x_n, \ldots \]

such that

\[ x_{n+1} = F(x_n), \text{ for } n \geq 0. \]

That is, the orbit of \( x_0 \) under \( F \) is the sequence of iterates

\[ x_0, F(x_0), F^2(x_0), F^3(x_0), \ldots, F^n(x_0), \ldots \]

\( x_0 \) is called the seed of the orbit. The changing values in an orbit represent a dynamical system. These are the kinds of dynamical systems we shall investigate in this course. At first glance such systems might seem simple, but their dynamics can turn out to be surprisingly complicated!

Example 1

Let \( F(x) = \sqrt{x} \). If \( x_0 = 256 \) then \( x_1 = \sqrt{256} = 16 \), \( x_2 = \sqrt{16} = 4 \); the first 8 terms in the orbit are

\[ 256, 16, 4, 2, 1.414213562, 1.189207115, 1.090507733, 1.044273783 \]

In this example, the orbit tends to the point 1; that is

\[ \lim_{n \to \infty} x_n = \lim_{n \to \infty} 256^{1/(2^n)} = 256^0 = 1. \]

If \( x_0 = .1 \), then the orbit under \( F \) also tends to 1:

\[ .1, .3162277660, .5623413252, .7498942093, .8659643233, .9305720409 \]

since \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} (.1)^{1/(2^n)} = (.1)^0 = 1. \)
Example 2

Let \( C(x) = \cos(x) \), with \( x \) in radians. Let \( x_0 = .5 \). Then the first six terms of the orbit under \( C \) are:

\[
.5, .8775825619, .6390124942, .8026851007, .6947780268, .7681958313
\]

Check that, to 10 digits,

\[
x_{53} = 0.7390851334, x_{54} = 0.7390851331, x_{55} = 0.7390851333,
\]

and

\[
x_{56} = 0.7390851332, x_{57} = 0.7390851332.
\]

So it appears that the orbit tends to a value of 0.7390851332, to 10 digits.

Example 3

Let \( F(x) = x^2 - 1 \), let \( x_0 = 1/2 \). The exact values of the first seven terms in the orbit of \( x_0 \) under \( F \) are:

\[
\frac{1}{2}, \frac{3}{4}, \frac{7}{16}, \frac{207}{256}, \frac{22687}{65536}, \frac{3780267327}{4294967296}, \frac{4156323010125826687}{18446744073709551616}
\]

Obviously exact calculations are cumbersome, even with a computer. Correct to 10 digits, the first 20 terms of the orbit are

\[
.5, -.75, -.4375, -.80859375, -.3461761475, -.8801620749, \\
n-.2253147219, -.9492332761, -.0989561875, -.9902076730, \\
n-.0194887643, -.9996201881, -.0007594795, -.9999994232, \\
n-.00000001536, -1, 0, -1, 0, -1
\]

In this example, the orbit ends up cycling between \(-1\) and 0.
Some Observations

- Calculating orbits requires lots of computation! At the least, you need a calculator. Better: use a computer algebra system, like Maple or Mathematica. Best: program a computer yourself. See Devaney’s algorithms in Appendix B.

- I will use Maple for my calculations. Warning: even with a computer or a computer algebra system, you will have to be careful! Computers have limited storage, so working with very small numbers can lead to ‘round-off’ problems and produce false results.

- The fact that so much computation power is required to investigate orbits is one reason that this area of mathematics only came into its own recently, with the advent of cheap, powerful computers.

Some Maple Functions

I will do most of my calculations with Maple 10. Here are some functions I have defined, specifically for Chapters 3 through 8:

```maple
orbit := proc(expression, seed, iterations)
local z, i, F; z := lhs(seed);
F := unapply(expression, z);
[seq((F@@i)(rhs(seed)), i=0..iterations)];
end:
```

This mini-program assumes the expression is $F(x)$, the seed is $x = x_0$, and iterations is a number $n$. Then orbit produces the sequence

$$x_0, F(x_0), F^2(x_0), \ldots, F^n(x_0).$$
decimalorbit := proc(expression, seed, iterations) local z, i, F; z := lhs(seed); F := unapply(expression, z); [seq(evalf((F@@i)(rhs(seed))),i=0..iterations)]; end:

The difference between orbit and decimalorbit is that the former calculates exact values of the iterates, but the latter calculates decimal approximations of the iterates, always using 10 digit accuracy.

plotorbit := proc(expression, seed, iterations) local z, F; z := lhs(seed); F := unapply(expression, z); plot([seq([i,(F@@i)(rhs(seed))],i=0..iterations)], style=POINT) end:

This mini-program plots the $n + 1$ points

$$(k, F^k(x_0)), \text{ for } k = 0, 1, \ldots, n;$$

this is one way to visualize the dynamics of the orbit.
cobweb :=
proc(expression,plotinterval,seed,iterations)
local z, i, j, F, horizontals, verticals, graph, line; z := lhs(seed); F := unapply(expression, z);
horizontals := plot([[seq([(F@@i)(rhs(seed)),(F@@(i+1))(rhs(seed))], [(F@@(i+1))(rhs(seed)),(F@@(i+1))(rhs(seed))]],i=0..iterations)], color=blue);
verticals := plot([[seq([(F@@j)(rhs(seed)),(F@@j)(rhs(seed))], [(F@@j)(rhs(seed)),(F@@(j+1))(rhs(seed))]],j=0..iterations)], color=blue);
graph := plot(expression,plotinterval,color=red);
line := plot(x,plotinterval,color=green);
plots[display](horizontals,verticals,graph,line);
end:

Maple Worksheets

cobweb is another mini-program designed to visualize the dynamics of the first \( n + 1 \) iterates of the orbit of the seed \( x = x_0 \) under the expression \( F(x) \), with respect to the graphs of

\[
y = F(x) \text{ and } y = x
\]
on the interval \([a, b]\). Many of my examples in Chapters 3 through 8 will be illustrated by using the above functions in Maple Worksheets. However, frequently we will have to go from the computational to the theoretical, to explain and understand what the computations show.
Fixed Point

$x$ is called a fixed point of $F$ if

$$F(x) = x.$$  

If $x_0$ is a fixed point of $F$, then $x_1 = F(x_0) = x_0$; generally

$$x_n = F^n(x_0) = x_0.$$  

Thus the orbit of $x_0$ under $F$ is a constant sequence; we also say the orbit is a fixed point, or that the orbit is fixed. In this case the dynamics is very simple to describe: nothing changes.

How To Find Fixed Points

To find the fixed points of $F$ you have to solve the equation $F(x) = x$ for $x$. This may or may not be possible.

1. Let $F(x) = x^2$. Then $F(x) = x \iff x^2 = x \iff x = 0$ or $1$.
2. Let $S(x) = \sin x$. Then $S(x) = x \iff \sin x = x \iff x = 0$.
3. Let $C(x) = \cos x$. Then $C(x) = x \iff \cos x = x \iff x = \pi$.
4. Let $F(x) = \sqrt{x}$. Then $F(x) = x \iff \sqrt{x} = x \iff x = 0$ or $1$.
5. Let $F(x) = x^3 - 1$. Then

$$F(x) = x \iff x^3 - 1 = x \iff x^3 - x - 1 = 0.$$  

But this cubic equation is hard to solve, even with the cubic formula.
Geometrical Description of Fixed Points

Geometrically, the fixed points of $F(x)$ are the intersection points of the graphs of the two curves

$$y = F(x) \text{ and } y = x.$$ 

The case $F(x) = x^2$ is illustrated to the right.

The other four cases from the previous slide are illustrated in the next slide.
Periodic Orbit, or Cycle

Another kind of orbit is the periodic orbit, or cycle. The point $x_0$ is periodic if

$$F^n(x_0) = x_0,$$

for some number $n$.

The least such $n$ is called the prime period of the orbit. (The prime period need not be a prime number!) The orbit of a periodic point $x_0$ under $F$ is just a repeating sequence of numbers

$$x_0, F(x_0), F^2(x_0), \ldots, F^{n-1}(x_0), x_0, F(x_0), F^2(x_0), \ldots, F^{n-1}(x_0), \ldots$$

The numbers

$$x_0, F(x_0), F^2(x_0), \ldots, F^{n-1}(x_0)$$

form a cycle, sometimes called an $n$-cycle. Note: every point on the cycle is also a periodic point of period $n$.

Example 1

1. $0$ lies on a cycle of prime period 2 for $F(x) = x^2 - 1$, since $F(0) = -1$ and $F(-1) = 0$. The orbit of 0 under $F$ is simply

$$0, -1, 0, -1, 0, -1, \ldots$$

2. $0$ lies on a 3-cycle for $F(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$:

$$F(0) = 1, F(1) = -\frac{3}{2} + \frac{5}{2} + 1 = 2, F(2) = -6 + 5 + 1 = 0.$$  

The orbit of 0 under $F$ is simply

$$0, 1, 2, 0, 1, 2, 0, 1, 2, \ldots$$
How To Find Periodic Points

If \( x \) is a periodic point of period \( n \) for \( F \) then \( F^n(x) = x \). This may be an extremely difficult equation to solve! For example, if \( F(x) = x^2 - 1 \), and \( x \) is a periodic point with period 3, then

\[
F^3(x) = x \quad \iff \quad x^8 - 4x^6 + 4x^4 - 1 = x
\]

\[
\iff \quad x^8 - 4x^6 + 4x^4 - x - 1 = 0
\]

\[
\iff \quad (x^2 - x - 1)(x^6 + x^5 - 2x^4 - x^3 + x^2 + 1) = 0
\]

One solution is \( \Phi = \frac{1 + \sqrt{5}}{2} \), which is actually a fixed point of \( F \).

Some of the other solutions may be periodic with prime period 3, but they are not so easy to find. Still with \( F(x) = x^2 - 1 \): solving for periodic points with period 5 would require solving a polynomial equation of degree 32. Good luck!

Eventually Fixed and Periodic Points

A point \( x_0 \) is called eventually fixed, or eventually periodic, if \( x_0 \) is itself not fixed or periodic, but some point on the orbit of \( x_0 \) is fixed or periodic.
Example 2

1. $-1$ is eventually fixed for $F(x) = x^2$, since $F(-1) = 1 \neq -1$ but $1$ is a fixed point of $F$.
2. $1$ is eventually periodic for $F(x) = x^2 - 1$, since $F(1) = 0$ and $0$ lies on a 2-cycle of $F$: $F(0) = -1, F(-1) = 0$.
3. $\sqrt{2}$ is also eventually periodic for $F(x) = x^2 - 1$ since $F(\sqrt{2}) = 1, F(1) = 0$ and $0$ is on a 2-cycle for $F$.
4. As we saw in Example 3 of Section 3.2, $x = 1/2$ is also eventually periodic for $F(x) = x^2 - 1$, but it takes 15 iterations to reach the 2-cycle:
   
   $F^{13}(0.5) = -0.9999994232, F^{14}(0.5) = -0.0000011536,$
   
   and
   
   $F^{15}(0.5) = -1$.

Here's `plotorbit` applied to the last example, to represent the orbit graphically:

```
plotorbit(x*x-1,x=.5,30);
```
Orbits With a Limit

If

$$\lim_{n \to \infty} F^n(x_0) = a,$$

then we say the orbit of $x_0$ under $F$ converges to $a$.

Note: if the orbit of $x_0$ under $F$ converges to $a$, then $a$ must be a fixed point of $F$. Why?

$$\lim_{n \to \infty} F^n(x_0) = a \Rightarrow F\left(\lim_{n \to \infty} F^n(x_0)\right) = F(a)$$

$$\Rightarrow \lim_{n \to \infty} F^{n+1}(x_0) = F(a)$$

$$\Rightarrow a = F(a)$$

Example 3

If $C(x) = \cos x$ and $x_0 = 0$, then the orbit of $x_0$ under $C$ converges to the unique fixed point of $\cos x$, which is about 0.7390851332, as we saw in Example 2 of Section 3.2.

plotorbit(cos(x), x=0, 30);
Other Orbits

Most orbits are not as simple as the ones described above. Moreover, the dynamics of orbits under $F$ may depend sensitively on $x_0$. For example, let $F(x) = x^2 - 2$. The orbit of $x_0 = 0$ under $F$ is eventually fixed: $F(0) = -2$, $F(-2) = 2$, $F(2) = 2$.

But consider the orbit if $x_0 = 0.1$; it never seems to become fixed at all. In the figure to the right, 100 iterations were calculated.

Or consider what happens if $x_0 = 0.01$ or $0.001$, using 100 iterations.

These orbits go all over the place! This kind of behaviour is called chaotic. Note: the points on these plots do not really agree with the figures on page 23 of the textbook because there the values are only calculated to 3 decimal places. Maple uses 10.
The Doubling Function

Define a function

\[ D : [0, 1) \longrightarrow [0, 1) \]

by

\[ D(x) = \begin{cases} 
2x & \text{if } 0 \leq x < \frac{1}{2} \\
2x - 1 & \text{if } \frac{1}{2} \leq x < 1 
\end{cases} \]

\(D\) is called the doubling function. \(D\) can be written in terms of one formula, \(D(x) = 2x - \lfloor 2x \rfloor\), where \(\lfloor x \rfloor\) is the greatest integer less than or equal to \(x\).

Example 1

1. 0 is the only fixed point of \(D\), because we restrict \(D\) to the interval \([0, 1)\).

2. \(\frac{1}{3}\) is on a 2-cycle:

\[ D\left(\frac{1}{3}\right) = \frac{2}{3}, \quad D\left(\frac{2}{3}\right) = \frac{4}{3} - 1 = \frac{1}{3} \]

3. \(\frac{1}{5}\) is on a 4-cycle:

\[ D\left(\frac{1}{5}\right) = \frac{2}{5}, \quad D\left(\frac{2}{5}\right) = \frac{4}{5}, \quad D\left(\frac{4}{5}\right) = \frac{3}{5}, \quad D\left(\frac{3}{5}\right) = \frac{1}{5} \]
Example 2

$D$ has many cycles. $1/9$ lies on a 6-cycle.

$$\text{orbit}(d(x), x=1/9, 12);$$

$$[1/9, 2/9, 4/9, 8/9, 7/9, 5/9, 1/9, 2/9, 4/9, 8/9, 7/9, 5/9, 1/9]$$

Example 3

$3/10$ is eventually periodic.

$$\text{orbit}(d(x), x=3/10, 10);$$

$$[3/10, 3/5, 1/5, 2/5, 4/5, 3/5, 1/5, 2/5, 4/5, 3/5, 1/5]$$

The orbit is eventually a 4-cycle.
Example 4

31/100 is also eventually periodic, but to a different cycle.

\[
\begin{bmatrix}
31 & 31 & 6 & 12 & 24 & 23 & 21 & 17 & 9 & 18 & 11 \\
22 & 19 & 13 & 1 & 2 & 4 & 8 & 16 & 7 & 14 & 3 & 6 \\
\end{bmatrix}
\]

The orbit is eventually a 20-cycle.

Example 5

29/100 is also eventually periodic, to the same 20-cycle as 31/100.

\[
\begin{bmatrix}
29 & 29 & 4 & 8 & 16 & 7 & 14 & 3 & 6 & 12 & 24 \\
23 & 21 & 17 & 9 & 18 & 11 & 22 & 19 & 13 & 1 & 2 & 4 \\
\end{bmatrix}
\]
Using Graphs to Analyze Orbits

One way to analyze an orbit is to consider the graphs of \( y = F(x) \) and \( y = x \). The intersection points of these two graphs are the fixed points of \( F \), and if the orbit of \( x_0 \) under \( F \) converges, then it will converge to a fixed point of \( F \). You can calculate the orbit of \( x_0 \) under \( F \) completely graphically:

1. Start at the point \((x_0, x_0)\) on the line with equation \( y = x \).
2. Go vertically to the graph of \( y = F(x) \); that will be the point \((x_0, F(x_0))\).
3. Now go horizontally to the graph of \( y = x \); that will be the point \((F(x_0), F(x_0)) = (x_1, x_1)\).
4. Repeat the process for the next iteration.

Example 1

Let \( S(x) = \sin x \). Here is the above process for a few iterations, starting with \( x_0 = 1 \): cobweb(sin(x),x=0..1.5,x=1,5):
Example 2

Here are two graphical representations of the orbit of $-2$ under $C(x) = \cos x$.

![Figure: cobweb(cos(x), x=-3..3, x=-2,10)](image1)

![Figure: plotorbit(cos(x), x=-2,10)](image2)

Example 3

Here’s the cobweb representation of the orbit of $x_0 = 0$ under $C(x) = \cos x$ near the fixed point:

![Figure: cobweb(cos(x), x=0, x=0.1,10)](image3)
Example 4

Here are two graphical representations of the orbit of 0.1 under $F(x) = 3.2x(1 - x)$.

Example 5

Here’s the cobweb representation of the orbit of $x_0 = .29$ under the doubling function:
What Is An Orbit Analysis?

An orbit analysis is a complete description of each orbit under $F$ for every possible value of $x_0$. This may not always be possible, but graphical analysis is a very convenient way to attempt an orbit analysis. For example, let $F(x) = x^3$. The only fixed points are 0, 1 and $-1$. These points determine 4 regions in which to pick $x_0$.

![Figure: $-1 < x_0 < 0$](image1)

![Figure: $0 < x_0 < 1$](image2)

![Figure: $x_0 < -1$](image3)

![Figure: $x_0 > 1$](image4)

If $|x_0| < 1$, then the orbit of $x_0$ under $F$ tends to 0. But if $|x_0| > 1$, then the orbit of $x_0$ under $F$ tends to plus or minus infinity.
What Is a Phase Portrait?

A phase portrait is another way of doing an orbit analysis. To portray the orbits of $x_0$ under $F$ you indicate on the real axis some of the points of the orbit indicating with arrows which way the orbit is going. So for example, consider $F(x) = x^3$, which has fixed points $x = 0, \pm 1$:

Another Example

Let $F(x) = x^2$. The fixed points are 0 and 1; $-1$ is an eventual fixed point. All other orbits will tend to 0 or $\infty$. 