This course covers systems of linear equations and Gaussian elimination, applications; vectors in $\mathbb{R}^n$, independent sets and spanning sets; linear transformations, matrices, inverses; subspaces in $\mathbb{R}^n$, basis and dimension; determinants; eigenvalues and diagonalization; systems of differential equations; dot products and orthogonal sets in $\mathbb{R}^n$; projections and the Gram-Schmidt process; diagonalizing symmetric matrices; least squares approximation. Includes an introduction to numeric computation in a weekly laboratory.

This is a first course in linear algebra. Although this material is intrinsically simpler than calculus, it may seem more difficult because it is more abstract than calculus. In Calculus there are only three basic concepts: limits, derivatives and integrals. In linear algebra there are at least 50 new concepts, some more important than others, but all are new ideas that you must assimilate! However, once you get used to the new concepts you will find that the computations in this course are more routine than those in calculus. Indeed, much of what we will do in MAT188H1F is of an algorithmic nature. The other aspect of MAT188H1F that makes it more abstract than calculus is that we will expect you to prove some things: proofs will be a part of lectures, homework and tests.

Section Instructors: by now you should be scheduled into one of the following Sections:

<table>
<thead>
<tr>
<th>LEC0101 D. Burbulla</th>
<th>LEC0102 S. Sorkhou</th>
<th>LEC0103 M. Greeff</th>
<th>LEC0104 S. Uppal</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEC0105 J. Sivaraman</td>
<td>LEC0106 L. Döppenschmitt</td>
<td>LEC0107 S. Cohen</td>
<td>LEC0108 D. Burbulla</td>
</tr>
</tbody>
</table>

Engineering Timetable: you should check the current timetable for Engineering math courses periodically, especially near the beginning of term since changes may occur.

Textbook: we are using an on-line open-source free textbook from Lyryx, namely

Nicholson's *Linear Algebra with Applications*.

All references with respect to the lecture schedule, suggested homework exercises, and the tutorial schedule are to this book. Nicholson’s book is non-standard in the sense that it covers eigenvalues and eigenvectors very early (Chapter 3) in the book. We used this version of Nicholson’s book last year, but this year we will follow a slightly different order of topics. In particular, we will spend the first three weeks covering the “simplest” material: solving systems of linear equations using Gaussian elimination; the algebra and geometry of vectors in $\mathbb{R}^n$; and the algebra of matrices. (This is why we jump around in Chapters 1, 2 and 4 during the first three weeks.) Note that the notation for a vector includes all of the following:

$X, \vec{x}, x, \overrightarrow{OX}$.

In terms of writing things on the board it is easiest to use the “arrow notation,” $\vec{x}$, for a vector, but you should be familiar with all the different notations. In addition you should look over this [general advice](#) (GA) for first year math students.

Marking Scheme: Practicals: 15%; Math Self-Assessment Quizzes: 2%; Tutorial Quizzes: 8%; Assignments/Quizzes in Lectures: 5%; Term Test 1: 15%; Term Test 2: 15%; Final Exam: 40%
Math Self-Assessment Quizzes: a link to the Math Self-Assessment Quizzes was emailed to all incoming Core 8 and TrackOne students in August. The Math Self-Assessment Quizzes will count 2% towards your final mark in MAT188H1F.

Suggested Homework: the homework exercises listed in the weekly Lecture Schedule include all the exercises that have answers and/or solutions in the back of the book. Warning: some of the exercises, especially near the end of a section, are quite challenging!

Math Learning Centre PG101: this is a drop-in centre that you can visit to get help with your math problems. Any Teaching Assistant there should be able to help you, although it can get very busy. There will be specific MAT188H1F TA’s available at certain times, to be posted. Here is a link to the [Math Learning Centre website](https://www.math.toronto.edu).

Assignments/Quizzes in Lectures: these depend totally on your lecturer, and may range from written assignments in essay form to on-line multiple choice questions in Quercus. You must participate in your lecture section, as found on your timetable, to get credit for this work.

Tutorial Quizzes: the schedule for tutorial quizzes is below, in the Tutorial Schedule. There will be five quizzes in tutorial, of which the best four will count. You must write the quiz in your own tutorial, as listed on your timetable, for it to count. The actual content in each tutorial, and what a quiz covers, will be posted on the course webpage before each cycle of tutorials.

Practicals: each student in MAT188H1F will be scheduled into a weekly lab covering numerical methods using Matlab. The labs are coordinated by Prof Variawa. All inquiries related to the labs should be sent to mat188h1@gmail.com.

Term Test 1: a 100-minute term test is scheduled for Tuesday, Oct 1, between 1:15 and 2:55 PM in locations to be announced. This test will be the same for all students.

Term Test 2: a 100-minute term test is scheduled for Tuesday, Nov 12, between 1:15 and 2:55 PM in locations to be announced. This test will be the same for all students.

Final Exam: there will be a common final exam, 150 min long, during the exam period, Dec 6-20. This exam will be the same for all students.

Calculators: use of a Casio FX-991 or Sharp EL-520 calculator will be permitted during all quizzes, tests and exams. However, it is still your responsibility to explain your work. A correct answer with no justification will receive no marks.

Alternate Reference: here is another on-line open-source textbook which you may find of interest if you want to read an alternate approach to some of the material:

[Kuttler’s Linear Algebra, A First Course](https://www.math.toronto.edu)

Course Coordinator: D. Burbulla. Office: GB 149

email: burbulla@math.toronto.edu; office hours: MR 10; TF 11; MWR 3; T 1-4

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1 Not on Sep 10, Oct 1, Oct 15, Nov 12 or Nov 26
Lecture Schedule: the Engineering term consists of 13 weeks of classes spread over 14 calendar weeks. The first day of classes is Thursday, Sep 5 and the last day of classes is Wednesday, Dec 4. Each lecturer will cover the material in his or her own style. Not everything in the reference sections will necessarily be covered in lectures. Although some material may be review of high school material for some of you, in lectures we shall assume that everything in the course is new for everybody. Instructors may become slightly ahead or behind schedule. In Week 6 six lecture sections will miss a class due to the Thanksgiving holiday, so less material is scheduled for Week 6.

First Class, Sep 5 or 6: course orientation.

Week 1, Sep 9–13: Systems of Linear Equations: solving a system of linear equations using elementary operations; the augmented matrix of a system of linear equations; row operations; row equivalent matrices; row echelon form of a matrix; reduced row echelon form of a matrix; Gaussian elimination; consistent systems and unique solutions; the rank of a matrix; homogeneous systems of linear equations; trivial and non-trivial solutions; linear combinations of solutions; basic solutions; some applications of systems of linear equations.

Ref: Sec 1.1, 1.2, 1.3, 1.4, 1.6.

Homework: Sec 1.1 #1(b), 4, 7(b), 10(b), 17, 19; Sec 1.2 #1(d),(f), 2(b), 3(b),(d), 5(d),(f),(h), 8(b), 9(b),(f), 12, 18; Sec 1.3 #1, 2(d), 3(b),(d), 5(b),(d), 7(b),(d), 11; Sec 1.4 #1, 3; Sec 1.6 #2, 4.

Week 2, Sep 16–20: Introduction to Vectors: vectors and lines in $\mathbb{R}^2$ or $\mathbb{R}^3$; dot products; properties of the dot product; angle between vectors; orthogonal vectors; projections; vector and normal equations of a plane; cross products (without determinants); minimum distance problems.

Ref: Sec 4.1, 4.2.

Homework: Sec 4.1 #1(b),(d),(f), 2(b), 6(b), 7(b),(d), 9(b),(d), (f), 10(b), 11(b), 17(b), 20(b), 21, 22(b),(d),(f), 23(b), 24(b),(d), 29; Sec 4.2 #2(b),(d),(f), 4(b), 6, 8(b), 10(b),(d), 11(b),(d), 12(b), 13(b),(d), 14, 15(b),(d),(f), 16(b), 19(b), 23(b), 24(b),(d), 34, 37, 38.

Week 3, Sep 23–27: More on Vectors: more properties of the cross product; vectors in $\mathbb{R}^n$. Algebra of Matrices: addition, scalar multiplication and transposition of matrices; symmetric matrices.

Ref: Sec 4.3, 5.3 beginning, 2.1

Homework: Sec 4.3 #2, 3(b), 4, 5, 6(b), 8, 10; Sec 5.3 #2, 5(b), 6(d), 12, 13, 14;
Sec 2.1 #1(b), 2(h), 6(b), 8(b), 13, 14(d), 15(d), 19, 20, 21.

Week 4, Sep 30–Oct 4: Vectors and Matrices: the algebra of vectors and matrices; systems of equations and matrix-vector multiplication; introduction to matrix transformations; matrix multiplication; composition of matrix transformations.

Ref: Sec 2.2, 2.3

Homework: Sec 2.2 #1(b), 2(b), 3(d), 4, 5(b),(d), 6, 8(b), 10, 11, 12, 13, 18;
Sec 2.3 #1(b),(d),(f),(h),(j), 3(b), 4(b), 6(b), 7(b), 8(b), 16(d), 20, 22(b), 27, 32, 34.


Ref: Sec 2.4, 2.6, (2.9 optional)

Homework: Sec 2.4 #2(d),(h),(i), 3(b),(d), 4(b), 5(b),(d),(h), 9, 12, 15(b), 16, 24(b), 25(b), 33(b), 34, 39(b); Sec 2.6 #1(b), 3, 4, 7(b), 8, 12, 14, 18.
Week 6, Oct 15-18: Linear operators on $\mathbb{R}^3$: more about projections, reflections and rotations.

Ref: Sec 4.4 beginning

Homework: Sec 4.4 #1, 2(b), 3(b),(d),(f), 6


Ref: Sec 3.1, 3.2, for the most part without proofs; Sec 4.3 rest; Sec 4.4 rest

Homework: Sec 3.1 #1(d),(f),(j),(n), 5(b), 6(b), 8(b), 9, 15(b), 16(b),(d), 26;
Sec 3.2 #2(b),(d),(f), 3(b), 4(b), 6(b), 10, 16, 20.

Week 8, Oct 28–Nov 1: Introduction to eigenvalues and eigenvectors: definitions and examples; the characteristic polynomial of a matrix; matrix diagonalization; powers of a matrix; systems of linear differential equations. Optional: linear dynamical systems or linear recurrences.

Ref: Sec 3.3, 3.5

Homework: Sec 3.3 #1(b),(d),(f),(h), 2(b), 8(b), 9, 12, 14, 20(b), 23, 27; Sec 3.5 #1(b),(d), 6

Week 9, Nov 4–8: The Vector Space $\mathbb{R}^n$: subspaces; spanning sets; linearly independent sets.

Ref: Sec 5.1, 5.2 beginning

Homework: Sec 5.1 #1(b),(d),(f), 2(b),(d), 3(b), 8, 10, 12, 13, 14, 15(b), 16, 17(a), 22;
Sec 5.2 #1(b),(d), 2(b),(d), 3(b),(d), 4(b),(d),(f), 5(b), 6(b),(d),(f), 7, 8, 10, 12, 14, 15, 17(b)

Week 10, Nov 11-15: More on the Vector Space $\mathbb{R}^n$: basis and dimension of a subspace; orthogonality and independence; subspaces determined by a matrix

Ref: Sec 5.2 rest, Sec 5.3 rest, Sec 5.4 beginning

Homework: Sec 5.3 #1(b), 3(b),(d), 4(b), 7, 15;
Sec 5.4 #1(b),(d), 2(b),(d), 3, 4, 5, 7(b), 9, 10, 15, 18


Ref: Sec 5.4 rest, 5.5, 5.6

Homework: Sec 5.5 #1(b),(d),(f), 3(b),(d), 4(b),(d), 7, 10, 13(b), 16;
Sec 5.6 #1(b), 2(b),(d), 3(b)

Week 12, Nov 25-29: Orthogonal complements and projections; the Gram-Schmidt algorithm.

Ref: Sec 8.1 without Theorem 4

Homework: Sec 8.1 #2(b),(d),(f), 3, 4(b),(d), 5, 10, 11, 16

Week 13, Dec 2–4: Symmetric matrices and orthogonal diagonalization. Review.

Ref: Sec 8.2

Homework: Sec 8.2 #1(f),(h), 5(b),(d),(f), 6, 15, 18, 19, 23, 24(a),(b)
\textbf{Tutorial Schedule:} each tutorial will meet 12 times during the term. Tutorials begin on Monday, Sep 9 and end on Monday, Dec 2. The University is closed on Thanksgiving Monday, Oct 14. So the first five tutorial cycles run from Monday, Sep 9 to Friday, Oct 11; the last seven tutorial cycles run from Tuesday, Oct 15 until Monday, Dec 2. Tutorials are roughly one week behind the lecture schedule. The actual content in each tutorial, and what a quiz covers, will be posted on the course webpage before each cycle of tutorials. To get marks for tutorial quizzes and/or assignments, you must attend your own tutorial. Only the best 4 of 5 tutorial quizzes will be counted.

1. \textbf{Sep 9–13:} general advice; proof.  
   Ref: GA and Appendix B

2. \textbf{Sep 16–20:} Quiz 1  
   Ref: Sec 1.1, 1.2, 1.3, 1.4, 1.6

3. \textbf{Sep 23–27:} Quiz 2  
   Ref: Sec 4.1, 4.2

4. \textbf{Sep 30–Oct 4:}  
   Ref: Sec 4.3, 5.3, 2.1

5. \textbf{Oct 7–11:} Quiz 3  
   Ref: Sec 2.2, 2.3

6. \textbf{Oct 15–21:}  
   Ref: Sec 2.4, 2.6

7. \textbf{Oct 22–28:} Quiz 4  
   Ref: Sec 4.4

8. \textbf{Oct 29–Nov 4:}  
   Ref: Sec 3.1, 3.2, 4.3, 4.4

9. \textbf{Nov 5–11:}  
   Ref: Sec 3.3, 3.5

10. \textbf{Nov 12–18:}  
    Ref: Sec 5.1, 5.2

11. \textbf{Nov 19–25:} Quiz 5  
    Ref: Sec 5.3, 5.4

12. \textbf{Nov 26–Dec 2:}  
    Ref: Sec 5.5, 5.6
Learning Outcomes (short version):

1. Write well-written, well-explained solutions to given problems using correct notation
2. Apply the algorithms of linear algebra to specific examples and to practical problems
3. Solve geometric problems in 3 dimensions using properties of vectors and determinants
4. Prove statements about matrices & vectors by using properties of matrices and vectors
5. Apply methods of linear algebra to analyze a real-world situation without cues

Learning Outcomes (long version):

• Given a problem, be it as simple as a calculation or as complicated as a long, involved word problem, students should be able to write a well-organized solution that defines any variables used, describes any assumptions made, includes diagrams that illustrate the connection between variables, uses correct mathematical notation, and provides full explanation of all the steps involved. In particular, students should be able to give complete solutions to the following types of problems:
  1. any word problem that can be reduced to a system of linear equations
  2. any word problem that can be reduced to a system of linear differential equations

• In the real vector space $\mathbb{R}^n$, students should be able to
  1. define a linear combination of $m$ vectors in $\mathbb{R}^n$
  2. define and calculate the dot product of two vectors
  3. define and calculate the length of a vector
  4. define and calculate the projection of one vector onto another non-zero vector
  5. define and calculate the cross product of two vectors in $\mathbb{R}^3$
  6. calculate the area of the parallelogram determined by two vectors in $\mathbb{R}^3$
  7. calculate the volume of the parallelepiped determined by three vectors in $\mathbb{R}^3$
  8. solve geometric problems in $\mathbb{R}^3$ that involve finding the minimum distance
     (i) from a line to a point not on the line
     (ii) from a plane to a point not on the plane
     (iii) between two skew lines, or between two parallel lines
  9. prove statements about vectors, using properties of vector addition, scalar multiplication, the length of a vector, the dot product, projections, and the cross product

• Given a system of $m$ linear equations in $n$ variables, students should be able to
  1. write down the augmented matrix of the system
  2. solve the system by using Gaussian elimination

• Given a subset $S$ of $\mathbb{R}^n$ students should be able to determine
  1. whether $S$ is linearly independent or not
  2. whether $S$ is orthogonal or not
3. whether \( S \) is orthonormal or not
4. whether \( S \) is a subspace of \( \mathbb{R}^n \) or not

- Given a subspace \( S \) of \( \mathbb{R}^n \) students should be able to find
  1. a spanning set for \( S \)
  2. a basis for \( S \)
  3. an orthogonal basis for \( S \), using the Gram-Schmidt algorithm, if necessary
  4. the dimension of \( S \)
  5. a basis for \( S^\perp \), the orthogonal complement of \( S \)
  6. the dimension of \( S^\perp \)
  7. the projection of a vector \( \vec{x} \) onto \( S \), and the projection of \( \vec{x} \) onto \( S^\perp \)

- Given matrices of appropriate sizes, students should be able to
  1. add them, subtract them, multiply them, calculate their transposes, and multiply them by a scalar
  2. solve problems and prove statements about matrices, using properties of matrix addition, matrix multiplication, and transposition

- Given an \( m \times n \) matrix \( A \) students should be able to
  1. calculate the reduced row echelon form of \( A \)
  2. find the rank of \( A \)
  3. find a basis for, and dimension of, the row space of \( A \), the column space of \( A \), and the null space of \( A \)
  4. state the Rank-Nullity Theorem
  5. solve problems and prove statements about matrices, using the Rank-Nullity Theorem

- Given an \( n \times n \) matrix \( A \) students should be able to
  1. calculate \( \det(A) \) by using suitable row and column operations, plus cofactor expansions
  2. determine if \( A \) is invertible, and if it is calculate \( A^{-1} \), the inverse of \( A \)
  3. solve problems and prove statements about determinants, using properties of determinants
  4. determine whether \( A \) is orthogonal or not
  5. state and make use of various properties of \( A \) that are equivalent to the statement “\( A \) is invertible”
  6. solve the matrix equation \( A \vec{x} = \vec{b} \) for \( \vec{x} \) by using \( A^{-1} \) if \( A \) is invertible
  7. define an eigenvalue and an eigenvector of \( A \)
  8. calculate the characteristic polynomial of \( A \)
  9. find the eigenvalues of \( A \) and a basis for each eigenspace of \( A \)
 10. diagonalize \( A \), or explain why it is not diagonalizable
 11. define what it means for the \( n \times n \) matrix \( B \) to be similar to \( A \)
 12. calculate powers of \( A \) if \( A \) is diagonalizable
13. solve the linear system of differential equations, \( \frac{d\vec{y}}{dt} = A\vec{y} \), for \( \vec{y} \) if \( A \) is diagonalizable

14. determine whether \( A \) is symmetric or not, and if it is, orthogonally diagonalize \( A \)

15. solve problems and prove statements about matrices, using properties of eigenvalues and eigenvectors

- Given a set of data points in \( \mathbb{R}^2 \), students should be able to find the least squares approximation linear, quadratic (or other) model that best fits the data, by making use of the normal equations

- Given a function \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \), students should be able to determine whether or not \( T \) is a linear transformation

- Given a linear transformation \( L : \mathbb{R}^m \rightarrow \mathbb{R}^n \) students should be able to

  1. calculate the standard matrix of \( L \)
  2. calculate \( L + M \), \( aL \) and \( L \circ N \), where \( a \) is a scalar in \( \mathbb{R} \), and \( M \) and \( N \) are other linear transformations
  3. calculate the standard matrices of \( L + M \), \( aL \) and \( L \circ N \) in terms of the matrices of \( L, M \) and \( N \)

- Students should be able to solve problems and prove statements about linear transformations, using properties of linear transformations

- Given a linear transformation \( L : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) students should be able to

  1. draw the image of the unit square under \( L \) and calculate the area of the image of the unit square in terms of the determinant of the matrix of \( L \)
  2. interpret geometrically the action of \( L \), if \( L \) is a dilation, a stretch, a shear, a projection, a rotation or a reflection in a line
  3. calculate the area of the image of a region \( \mathcal{R} \) under \( L \) in terms of the area of \( \mathcal{R} \) and the determinant of the matrix of \( L \)

- Given a linear transformation \( L : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) that represents a projection onto a subspace of \( \mathbb{R}^3 \), or a reflection in a plane in \( \mathbb{R}^3 \), students should be able to calculate the matrix of \( L \)

- Matlab stuff . . .