Functions Test Solutions.

Answers to the Test:

Solutions and Comments:

1. There are exactly three real solutions to the equation $x^3 = 1 - x$; this is False.

Any solution to this equation would represent an intersection point on the graphs of $y = x^3$ and $y = 1 - x$. The graphs are to the right. There is only one intersection point. So the solution to the equation $x^3 = 1 - x$ is somewhere between $x = 0$ and $x = 1$. Note: there are three solutions to the equation if you permit complex solutions.

2. There are exactly three real solutions to the equation $3^x = 4x^2$; this is True.

But it’s not obvious. You can use the same approach as in the previous question. The graphs of $y = 3^x$ and $y = 4x^2$ are to the right: the quadratic is in green; the exponential is in red. You can see that there are three intersection points.

3. The range of the graph with equation $x^{2/3} + y^{2/3} = 4$ is $-8 \leq y \leq 8$; this is True.

$$0 \leq x^{2/3} = 4 - y^{2/3} \Rightarrow y^{2/3} \leq 4 \Rightarrow y^2 \leq 64 \Rightarrow |y| \leq 8 \Rightarrow -8 \leq y \leq 8.$$
4. If a sequence $F_n$ is defined by $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, then $F_6 = 6$; this is False.

Aside: this is the Fibonacci sequence. Compute:

$$F_2 = F_1 + F_0 = 1; F_3 = F_2 + F_1 = 2; F_4 = F_3 + F_2 = 3; F_5 = F_4 + F_3 = 5; F_6 = F_5 + F_4 = 8.$$ 

5. The inverse of the function $f(x) = \frac{2x + 3}{x - 5}$ is $f^{-1}(x) = \frac{5x + 3}{x - 2}$.

To find the inverse of $y = f(x)$ interchange $x$ and $y$ and solve for $y$:

$$x = \frac{2y + 3}{y - 5} \Rightarrow x(y - 5) = 2y + 3$$

$$\Rightarrow xy - 5x = 2y + 3$$

$$\Rightarrow xy - 2y = 5x + 3$$

$$\Rightarrow y(x - 2) = 5x + 3$$

$$\Rightarrow y = \frac{5x + 3}{x - 2}$$

6. The number of asymptotes to the graph of $f(x) = \frac{x^2 + 1}{x + 1}$ is 2.

$$f(x) = \frac{x^2 + 1}{x + 1} = x - 1 + \frac{2}{x + 1},$$

by long division. So

$$x = -1$$

is a vertical asymptote to the graph of $f$,

and

$$y = x - 1$$

is a slant asymptote to the graph of $f$. See the graph to the right.

7. If $g(x) = \frac{1}{x}$ and $h \neq 0$, then $\frac{g(x + h) - g(x)}{h} = \frac{-1}{x(x + h)}$. 


Straight simplification:

\[
\frac{g(x + h) - g(x)}{h} = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \frac{-h}{hx(x + h)} = \frac{-1}{x(x + h)}
\]

8. Let \( f(x) = 3x - 2 \), let \( g(x) = x^2 - 1 \). Then \( f(g(x)) = 3x^2 - 5 \).

\[
f(g(x)) = f(x^2 - 1) = 3(x^2 - 1) - 2 = 3x^2 - 3 - 2 = 3x^2 - 5.
\]

9. If \( |2x - 4| \leq |x + 3| \), then \( \frac{1}{3} \leq x \leq 7 \).

One way to solve this is to plot graphs. To the right, the red graph is the graph of

\[
y = |2x - 4|
\]

and the green graph is the graph of

\[
y = |x + 3|.
\]

You can see that the red graph is below the green graph for

\[
\frac{1}{3} \leq x \leq 7.
\]

Or you can solve the inequality algebraically, using \( |z|^2 = z^2 \) to eliminate the absolute value signs.

\[
|2x - 4| \leq |x + 3| \iff |2x - 4|^2 \leq |x + 3|^2 \\
\iff (2x - 4)^2 \leq (x + 3)^2 \\
\iff 4x^2 - 16x + 16 \leq x^2 + 6x + 9 \\
\iff 3x^2 - 22x + 7 \leq 0 \\
\iff (3x - 1)(x - 7) \leq 0 \\
\iff \frac{1}{3} \leq x \leq 7
\]