Second-Order Differential Equations

Chapter Preview  In Chapter 8, we introduced first-order differential equations and illustrated their use in describing how physical and biological systems change in time or space. As you will see in this chapter, second-order differential equations are equally applicable and are widely used for similar purposes in many disciplines. After presenting some fundamental concepts that underlie second-order linear equations, we turn to linear constant-coefficient equations, which happen to be among the most applicable of all differential equations. After learning how to solve these equations and their associated initial value problems, we discuss a few of the many mathematical models based on second-order equations. The chapter closes with a look at transfer functions, which are used to analyze and design mechanical and electrical oscillators.

16.1 Basic Ideas

Much of what you learned about first-order differential equations in Chapter 8 will be useful in the study of second-order equations. Once again, you will see the idea of a general solution, which is an entire family of functions that satisfy the equation. However, many of the methods used to find general solutions of first-order equations do not work for second-order equations. As a result, much of the chapter is devoted to developing new solution methods. At the same time, we highlight many applications of second-order equations.

A Quick Overview

Perhaps the most common source of second-order differential equations is Newton’s second law of motion, which governs the motion of everyday objects (for example, planets, billiard balls, and raindrops). Therefore, much of this chapter is devoted to developing mathematical formulations of systems that are in motion or that have time-dependent behavior. As you will see, a system may be a moving object such as a falling stone, a swinging pendulum, or a mass on a spring. Less obvious, a system may also be an electrical circuit that produces a radio signal, a boat in pursuit of a fleeing target, or the organs of a person assimilating a drug.

Here is an example of a system. Imagine a block of mass \( m \) hanging at rest from a solid support by a spring. If the block is displaced from its rest position and released, then it oscillates up and down along a line (Figure 16.1). We let \( y(t) \) be the position of the block relative to its rest position \( t \) time units after it is released. When the spring is stretched below the rest position, the position of the block \( y(t) \) is positive.
Newton’s second law for one-dimensional motion governs the motion of the block; it says that
\[ \frac{\text{mass \cdot acceleration}}{\text{mass}} = \frac{\text{sum of forces}}{\text{force}} \]

We know that the acceleration is \( a(t) = y''(t) \). Therefore, Newton’s second law takes the form
\[ my''(t) = F, \]
where the forces included in \( F \) (such as the restoring force of the spring, air resistance, and external forces) may depend on the time \( t \), the position \( y \), and the velocity \( y' \).

We will investigate the spring-block system in detail in Section 16.4. As you will see, a complete mathematical formulation of this system includes a differential equation, with all the relevant external forces, plus a set of initial conditions. The initial conditions specify the initial position and velocity of the block. A typical set of initial conditions has the form \( y(0) = A \), \( y'(0) = B \), where \( A \) and \( B \) are given constants.

This combination of a differential equation plus initial conditions is called an **initial value problem**. The goal of this chapter is to learn how to solve second-order initial value problems.

**Terminology**
Recall that the **order** of a differential equation is the highest order that appears on a derivative in the equation. This chapter deals with **linear** second-order equations of the form
\[ y''(t) + p(t)y'(t) + q(t)y(t) = f(t). \]  
(1)

In this equation, \( p, q, \) and \( f \) are specified functions of \( t \) that are continuous on some interval of interest that we call \( I \). The equation is linear because the unknown function \( y \) and its derivatives appear only to the first power, and not in products with each other, or as arguments of other functions. Equations that cannot be put in this form are **nonlinear**. Solving equation (1) means finding a function \( y \) that satisfies the equation on the interval \( I \).

Another useful distinction concerns the function \( f \) on the right side of equation (1). An equation in which \( f(t) = 0 \) on the interval of interest is said to be **homogeneous**. An equation in which \( f \) is not identically zero is **nonhomogeneous**.

**EXAMPLE 1** Classifying differential equations Classify the following differential equations that arise from Newton’s second law.

a. \( my'' = -0.001y' - 2.1y \) (This equation describes a block of mass \( m \) oscillating on a spring in the presence of friction.)

b. \( my'' = mg - 0.05(y')^2 \) (This equation describes an object of mass \( m \) falling in a gravitational field subject to air resistance, where \( g \) is the acceleration due to gravity.)

**SOLUTION**

a. Writing the equation in the form \( y'' + (0.001/m)y' + (2.1/m)y = 0 \), we see that it has the form given in (1). The term with the highest order derivative is \( y'' \); therefore, the equation is second order. It is linear because \( y \) and its derivatives appear only to the first power, and they do not appear in products or composed with other functions. It is a homogeneous equation because there is no term independent of \( y \) and its derivatives.

b. Writing the equation in the form \( y'' + (0.001/m)y' + (2.1/m)y = 0 \), we see that it has the form given in (1). The term with the highest order derivative is \( y'' \); therefore, the equation is second order. It is linear because \( y \) and its derivatives appear only to the first power, and they do not appear in products or composed with other functions. It is a homogeneous equation because there is no term independent of \( y \) and its derivatives.
b. As in part (a), the equation is second order. It is nonlinear because \( y' \) appears to the second power, and it is nonhomogeneous because the term \( mg \) is independent of \( y \) and its derivatives.

**Related Exercises 9–12**

### Quick Check 1
Classify these equations with respect to order, linearity, and homogeneity.

A: \( y' + 3y = 4t^2 \), B: \( y'' - 4y' + 2y = 0 \).

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### Homogeneous Equations and General Solutions

We now turn to second-order linear homogeneous equations of the form

\[ y'' + py' + qy = 0, \]

and see what it means for a function to be a solution of such an equation.

**Example 2** **Verifying solutions** Consider the linear differential equation

\[ t^2y'' - ty' - 3y = 0, \quad \text{for } t > 0. \]

a. Verify by substitution that the functions \( y = t^3 \) and \( y = \frac{1}{t} \) are solutions of the equation.

b. Verify by substitution that the function \( y = 100t^3 \) is a solution of the equation.

c. Verify by substitution that the function \( y = 6t^3 + \frac{8}{t} \) is a solution of the equation.

**Solution**

a. Substituting \( y = t^3 \) into the equation, we carry out the following calculations.

\[
\begin{align*}
  t^2(t^3)'' &- t(t^3)' - 3(t^3) \\
  y'' &- 6t \quad y' = 3t^2 \\
  y &- t^3 \\
  &- t^3(6 - 3 - 3) \\
  &- 0.
\end{align*}
\]

We see that \( y = t^3 \) satisfies the equation, for all \( t > 0 \). Substituting \( y = t^{-1} \) into the equation, we find that

\[
\begin{align*}
  t^2(t^{-1})'' &- t(t^{-1})' - 3(t^{-1}) \\
  y'' &- 2t^{-3} \quad y' = -t^{-2} \\
  y &- t^{-1} \\
  &- t^{-3}(2 + 1 - 3) \\
  &- 0.
\end{align*}
\]

The function \( y(t) = t^{-1} \) also satisfies the equation, for all \( t > 0 \).

b. Recall that \( (cy(t))^r = cy(t)^r \) for real numbers \( c \). So you might anticipate that multiplying the solution \( y(t) = t^3 \) by the constant 100 will produce another solution. A quick check shows that

\[
\begin{align*}
  t^2((100t^3)'' &- t(100t^3)' - 3(100t^3) \\
  y'' &- 200t^{-3} \quad y' = 300t^{-2} \\
  y &- 100t^3 \\
  &- 100(t^3(6 - 3 - 3)) \\
  &- 0.
\end{align*}
\]

The function \( y = 100t^3 \) is a solution. We could replace 100 by any constant \( c \) and the function \( y = ct^3 \) would also be a solution. Similarly, \( y = ct^{-1} \) is a solution, for any constant \( c \).
c. By parts (a) and (b), we know that $y = t^3$ and $y = t^{-1}$ are both solutions of the equation. Now we investigate whether a constant multiplied by one solution plus a constant multiplied by the other solution is also a solution. Substituting, we have

$$
\begin{align*}
(6t^3 + 8t)^{-1} &- (6t^3 + 8t^{-1})' - 3(6t^3 + 8t^{-1}) \\
y' &= 6t^2 - 8t^{-2} \\
y &= 6t^3 + 8t^{-1}
\end{align*}
$$

In this case, the sum of constant multiples of two solutions is also a solution, for any constants.

Example 2 raises some fundamental questions about linear differential equations and it gives some hints about answers. How many solutions does a second-order linear equation have? When can you multiply a solution by a constant (as in Example 2b) and produce another solution? When can you add two solutions (as in Example 2c) and get another solution? Focusing on homogeneous equations, the following theorem begins to answer these questions.

**THEOREM 16.1 Superposition Principle**

Suppose that $y_1$ and $y_2$ are solutions of the homogeneous second-order linear equation $y'' + py' + qy = 0$. Then the function $y = c_1y_1 + c_2y_2$ is also a solution of the homogeneous equation, where $c_1$ and $c_2$ are arbitrary constants.

**Proof:** We verify by substitution that the function $y = c_1y_1 + c_2y_2$ satisfies the equation.

$$
(c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2)
$$

$$
= c_1y_1'' + c_1py_1' + c_1qy_1 + c_2y_2'' + c_2py_2' + c_2qy_2
$$

$$
= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2)
$$

equals 0; $y_1$ is a solution

equals 0; $y_2$ is a solution

$$
= c_1 \cdot 0 + c_2 \cdot 0
$$

$$
= 0
$$

We have confirmed that $y = c_1y_1 + c_2y_2$ is a solution of the homogeneous equation when $y_1$ and $y_2$ are solutions.

A function of the form $c_1y_1 + c_2y_2$ is called a **linear combination** or superposition of $y_1$ and $y_2$. Theorem 16.1 says that linear combinations of solutions of a linear homogeneous equation are also solutions. This important property applies only to linear differential equations.

We now turn to the question of whether a linear combination such as $c_1y_1 + c_2y_2$ accounts for all the solutions of a homogeneous equation. The following definition is critical.

**DEFINITION** Linear Dependence/Independence of Two Functions

Two functions $\{f_1(t), f_2(t)\}$ are **linearly dependent** on an interval $I$ if one function is a nonzero constant multiple of the other function, for all $t$ in $I$, that is, for some nonzero constant $c$, $f_1(t) = cf_2(t)$, for all $t$ in $I$. Otherwise, $\{f_1(t), f_2(t)\}$ are **linearly independent** on $I$. 
For example, the functions \( \{ t, t^2 \} \) are linearly independent on any interval because there is no constant \( c \) such that \( t = ct^2 \), for all \( t \) in that interval (Figure 16.2a). Similarly, the functions \( \{ \sin t, \cos t \} \) are linearly independent on any interval, whereas the functions \( \{ e^t, 2e^t \} \) are constant multiples of each other and are linearly dependent on any interval (Figure 16.2b).

\[ y = t \text{ and } y = t^2 \text{ are linearly independent on any interval} \]

\[ y = e^t \text{ and } y = 2e^t \text{ are linearly dependent on any interval} \]

\[ \text{FIGURE 16.2} \]

Using the same argument, the following pairs of functions are linearly independent:

\[ \{ \sin at, \cos bt \} \text{ on } (-\infty, \infty), \text{ for real numbers } a \neq 0 \text{ and } b, \]

\[ \{ e^{at}, e^{bt} \} \text{ on } (-\infty, \infty), \text{ for real numbers } a \neq b, \]

\[ \{ t^p, t^q \} \text{ on } (0, \infty), \text{ for real numbers } p \neq q. \]

**An Aside** The concept of linear independence is important in many areas of mathematics and it applies to objects other than functions. More formally, a set of \( n \) functions \( \{ f_1(t), f_2(t), \ldots, f_n(t) \} \) is linearly dependent on an interval \( I \) if there are constants \( c_1, c_2, \ldots, c_n \), not all zero, such that

\[ c_1f_1(t) + c_2f_2(t) + \cdots + c_nf_n(t) = 0, \text{ for all } t \text{ in } I. \]

Equivalently, if one function in the set can be written as a linear combination of the other functions, then the functions are linearly dependent. If this identity holds only by taking \( c_1 = c_2 = \cdots = c_n = 0 \), then the functions are linearly independent.

For example, the functions \( \{ 1, t, t^2 \} \) are linearly independent, whereas the functions \( \{ t, t^2, 3t^2 - 2t \} \) are linearly dependent on \((-\infty, \infty)\). When \( n = 2 \), this more general definition reduces to the definition given above.

As stated in the following theorem, linear independence is the key to determining whether we have found all the solutions of a linear homogeneous differential equation.

**THEOREM 16.2**

If \( p \) and \( q \) are continuous on an interval \( I \), and \( y_1 \) and \( y_2 \) are linearly independent solutions of the linear homogeneous equation \( y'' + py' + qy = 0 \), then all solutions of the homogeneous equation can be expressed as a linear combination \( y = c_1y_1 + c_2y_2 \), where \( c_1 \) and \( c_2 \) are arbitrary constants.
If \( y_1 \) and \( y_2 \) are linearly independent solutions, the function \( y = c_1 y_1 + c_2 y_2 \), where \( c_1 \) and \( c_2 \) are arbitrary real constants, is called the general solution of the homogeneous equation; it represents all possible homogeneous solutions.

Notice the progression here. The general solution of a first-order differential equation involves one arbitrary constant; the general solution of a second-order equation involves two arbitrary constants; and the general solution of an \( n \)th-order equation involves \( n \) arbitrary constants.

**EXAMPLE 3**  General solutions

a. The functions \( \{ e^t, e^{t+2} \} \) are solutions of the equation \( y'' - y = 0 \), for \(-\infty < t < \infty \). If possible, find a general solution of the equation.

b. The functions \( \{ e^{4t}, e^{-4t} \} \) are solutions of the equation \( y'' - 16y = 0 \), for \(-\infty < t < \infty \). Show that \( y = \cosh 4t \) is also a solution.

**SOLUTION**

a. Noting that \( e^{t+2} = e^2 e^t \), we see that \( e^{t+2} \) is a constant multiple of \( e^t \) for all \( t \) in \(( -\infty, \infty ) \). Therefore, the functions \( \{ e^t, e^{t+2} \} \) are linearly dependent, and we cannot determine the general solution from this information alone. Another linearly independent solution is needed in order to write the general solution. (You can verify that \( e^{-t} \) is a second linearly independent solution.)

b. The functions \( \{ e^{4t}, e^{-4t} \} \) are linearly independent on \(( -\infty, \infty ) \) because there is no constant \( c \) such that \( e^{4t} = c e^{-4t} \) for all \( t \) in \(( -\infty, \infty ) \). Therefore, by Theorem 16.2 we can write all solutions of the homogeneous equation in the form \( c_1 e^{4t} + c_2 e^{-4t} \). For example, taking \( c_1 = c_2 = \frac{1}{2} \), we see that \( \cosh 4t = \frac{1}{2} e^{4t} + \frac{1}{2} e^{-4t} \) is also a solution.

**EXAMPLE 4**  An oscillator equation  The equation \( y'' + 9y = 0 \) describes the motion of an oscillator such as a block on a spring in the absence of external forces such as friction. The functions \( \{ \sin 3t, \cos 3t \} \) are solutions of the equation, for \(-\infty < t < \infty \). Find the general solution of the equation.

**SOLUTION**  The functions \( \{ \sin 3t, \cos 3t \} \) are linearly independent on \(( -\infty, \infty ) \) because it is not possible to find a constant \( c \) such that \( \sin 3t = c \cos 3t \), for all \( t \) in \(( -\infty, \infty ) \). Therefore, the general solution can be written in the form \( y = c_1 \sin 3t + c_2 \cos 3t \), where \( c_1 \) and \( c_2 \) are real numbers.

**Nonhomogeneous Equations and General Solutions**

We now shift our attention to linear nonhomogeneous equations of the form

\[
y''(t) + p(t)y'(t) + q(t)y(t) = f(t),
\]

where the function \( f \) is not identically zero on the interval of interest. As before, we assume that \( p, q, \) and \( f \) are continuous on some interval \( I \) of interest. Suppose for the moment that we have found a function that satisfies this equation. Such a solution is called a particular solution, and methods for finding particular solutions are discussed in Section 16.3.

**EXAMPLE 5**  Another oscillator equation  Building on Example 4, the equation \( y'' + 9y = 14 \sin 4t \) describes a spring-block system that is driven by an oscillatory external force \( f(t) = 14 \sin 4t \) in the absence of friction. Show that \( y_p = -2 \sin 4t \) is a particular solution of the equation.
16.1 Basic Ideas

SOLUTION Substituting \( y_p = -2 \sin 4t \) into the nonhomogeneous equation, we have

\[
y'' + 9y = (-2 \sin 4t)'' + 9(-2 \sin 4t) = -2(-16 \sin 4t) - 18 \sin 4t = 14 \sin 4t.
\]

Substitute \( y_p \). Simplify.

Therefore, \( y_p \) satisfies the nonhomogeneous equation and is a particular solution.

Related Exercises 27–30

Quick Check 3. Is \( y_p = -1 \) a particular solution of the equation \( y'' - y = 1 \)?

Our goal is to find the general solution of a given nonhomogeneous equation; that is, a family of functions, all of which satisfy the equation. Before doing so, we can answer an important practical question right now. How many particular solutions does one equation have? When do we stop looking? Theorem 16.3 provides the answers.

Theorem 16.3

If \( y_p \) and \( z_p \) are particular solutions of the nonhomogeneous equation \( y'' + py' + qy = f \), then \( y_p \) and \( z_p \) differ by a solution of the homogeneous equation.

Proof: Let \( w = y_p - z_p \) be the difference of two particular solutions and note that \( y_p \) and \( z_p \) both satisfy the nonhomogeneous equation. Substituting \( w \) into the differential equation, we find that

\[
w'' = w' + pqw = (y_p - z_p)'' + p(y_p - z_p)' + q(y_p - z_p)
\]

= \( y_p'' + py_p' + qy_p - (z_p'' + pzp' + qz_p) \) Regroup; identify particular solutions.

= \( f - f = 0 \).

The practical meaning of the theorem is that if you find one particular solution, then you can stop looking. Any two particular solutions must differ by a solution of the homogeneous equation, and solutions of the homogeneous equation already appear in the general solution.

We can now describe how to find the general solution of a nonhomogeneous equation: We find the general solution of the homogeneous equation \( c_1y_1 + c_2y_2 \) and add to it any particular solution.

Theorem 16.4

Suppose \( y_1 \) and \( y_2 \) are linearly independent solutions of the homogeneous equation \( y'' + py' + qy = 0 \), and \( y_p \) is any particular solution of the corresponding nonhomogeneous equation \( y'' + py' + qy = f \). Then the general solution of the nonhomogeneous equation is

\[
y = c_1y_1 + c_2y_2 + y_p,
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants.
Proof: Notice that because of Theorem 16.3, we can choose any particular solution to form the general solution. We verify by substitution that \( y = c_1 y_1 + c_2 y_2 + y_p \) satisfies the nonhomogeneous equation. Recall that \( y_1 \) and \( y_2 \) satisfy \( y'' + py' + qy = 0 \) and \( y_p \) satisfies \( y'' + py' + qy = f \).

\[
y'' + py' + qy = (c_1 y_1 + c_2 y_2 + y_p)'' + p(c_1 y_1 + c_2 y_2 + y_p)' + q(c_1 y_1 + c_2 y_2 + y_p)
\]

Substitute solution.

\[
= c_1 (y_1'' + py_1' + qy_1) + c_2 (y_2'' + py_2' + qy_2) + \left( \frac{y_p''}{f} + py_p' + qy_p \right)
\]

Rearrange terms.

\[
= 0 + 0 + f = f
\]

Identify solutions.

We see that the proposed general solution satisfies the nonhomogeneous equation, as claimed. Notice that general solution of the nonhomogeneous equation also has two arbitrary constants.

EXAMPLE 6  General solution of an oscillator equation  Find the general solution of the oscillator equation \( y'' + 9y = 14 \sin 4t \) (Example 5).

SOLUTION  By Example 4, two linearly independent solutions of the homogeneous equation are \( y_1 = \sin 3t \) and \( y_2 = \cos 3t \). Using Example 5, we know that a particular solution is \( y_p = -2 \sin 4t \). By Theorem 16.4, the general solution of the oscillator equation is

\[
y = c_1 \sin 3t + c_2 \cos 3t - 2 \sin 4t,
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

Initial Value Problems

As mentioned at the beginning of this chapter, mathematical models that involve differential equations often take the form of an initial value problem; that is, a differential equation accompanied by initial conditions. It turns out that with second-order equations, two initial conditions are needed to specify a solution to the initial value problem. Unless there is a good reason to do otherwise, we specify the initial conditions at \( t = 0 \). For equations that describe the motion of an object, the initial conditions give the initial position and velocity of the object. As shown in the next example, the two initial conditions are used to determine the two arbitrary constants in the general solution.

EXAMPLE 7  Solution of an initial value problem  Consider the spring-block system described in Example 6. If the block has an initial position \( y(0) = 4 \) and an initial velocity \( y'(0) = 1 \), the motion of the block is described by the initial value problem

\[
y'' + 9y = 14 \sin 4t \quad \text{Differential equation}
\]

\[
y(0) = 4, \quad y'(0) = 1. \quad \text{Initial conditions}
\]

Find the solution of the initial value problem.

SOLUTION  The general solution of the differential equation was found in Example 6:

\[
y = c_1 \sin 3t + c_2 \cos 3t - 2 \sin 4t.
\]
To determine the two arbitrary constants \( c_1 \) and \( c_2 \), we use the initial conditions. The first condition \( y(0) = 4 \) implies that
\[
y(0) = c_1 \sin(3 \cdot 0) + c_2 \cos(3 \cdot 0) - 2 \sin(4 \cdot 0) = c_2 = 4,
\]
and the constant \( c_2 = 4 \) is determined. Noting that
\[
y' = 3c_1 \cos 3t - 3c_2 \sin 3t - 8 \cos 4t,
\]
the second condition \( y'(0) = 1 \) implies that
\[
y'(0) = 3c_1 \cos(3 \cdot 0) - 3c_2 \sin(3 \cdot 0) - 8 \cos(4 \cdot 0) = 3c_1 - 8 = 1;
\]
it follows that \( c_1 = 3 \). Having determined the two arbitrary constants in the general solution, the solution of the initial value problem is
\[
y = 3 \sin 3t + 4 \cos 3t - 2 \sin 4t.
\]
In practice, it is advisable to check that this function does everything it is supposed to do: It must satisfy the differential equation and both initial conditions.

**Figure 16.3** shows that the solution to the initial value problem (in red) is one of infinitely many functions in the general solution. It is the only one that satisfies the initial conditions.

> **QUICK CHECK 5** The general solution of an equation is \( y = c_1 \sin t + c_2 \cos t \). Find the constants \( c_1 \) and \( c_2 \) such that \( y(0) = 1, y'(0) = 0 \).

### Theoretical Matters

We close with two important questions. We can provide answers, but rigorous proofs go beyond the scope of this discussion and are generally given in advanced courses.

The first question concerns solutions of initial value problems. Given an initial value problem such as that in Example 7, when can we expect to find a unique solution? An answer is given in the following theorem.

#### Theorem 16.5 Solutions of Initial Value Problems

Suppose the functions \( p, q, \) and \( f \) are continuous on an open interval \( I \) containing the point \( 0 \). Then the initial value problem
\[
y''(t) + p(t)y'(t) + q(t)y(t) = f(t)\\y(0) = A, \ y'(0) = B,
\]
where \( A \) and \( B \) are given, has a unique solution on \( I \).
The conditions of this theorem, namely continuity of the coefficients $p$, $q$, and $f$ on the interval of interest, guarantee the existence and uniqueness of solutions of initial value problems on same interval. These conditions are satisfied by the equations we consider in this chapter.

The second question concerns general solutions. All the examples of this section have demonstrated that second-order linear homogeneous equations have two linearly independent solutions, which comprise the general solution. Is this observation always true? The following theorem gives an affirmative answer under appropriate conditions.

**THEOREM 16.6  ** Linearly Independent Solutions
Suppose the functions $p$ and $q$ are continuous on an open interval $I$. Then the homogeneous equation
\[ y''(t) + p(t)y'(t) + q(t)y(t) = 0 \]
has two linearly independent solutions $y_1$ and $y_2$, and the general solution on $I$ is $y = c_1y_1 + c_2y_2$, where $c_1$ and $c_2$ are arbitrary constants.

These theorems claim the existence of solutions, but they don’t say a word about how to find solutions. We now turn to the practical matter of actually solving differential equations.

**SECTION 16.1 EXERCISES**

**Review Questions**

1. Describe how to find the order of a differential equation.
2. How do you determine whether a differential equation is linear or nonlinear?
3. What distinguishes a homogeneous from a nonhomogeneous differential equation?
4. Give a general form of a second-order linear nonhomogeneous differential equation.
5. How do you determine whether two functions are linearly dependent on an interval?
6. How many linearly independent functions appear in the general solution of a second-order linear homogeneous differential equation?
7. Explain how to find the general solution of a second-order linear nonhomogeneous differential equation.
8. Explain the steps used to find the solution of an initial value problem that involves a second-order linear nonhomogeneous differential equation.

**Basic Skills**

9–12. Classifying differential equations Determine the order of the following differential equations. Then state whether they are linear or nonlinear, and whether they are homogeneous or nonhomogeneous.

9. $y'' - 4y' + 2y = 10r^2$
10. $y' = 2y^3 - 4t$
11. $y'' - 3yy' - y = e^t$
12. $z'' + 16z = 0$

13–22. Verifying solutions Verify by substitution that the following equations are satisfied by the given functions. Assume that $c_1$ and $c_2$ are arbitrary constants.

13. $y'' - 4y = 0; \text{ solution } y = 3e^{2t} - 5e^{-2t}$
14. $y'' + 16y = 0; \text{ solution } y = 10 \sin 4t - 20 \cos 4t$
15. $y'' - 9y = 18t; \text{ solution } y = 4e^{3t} + 3e^{-3t} - 2t$
16. $y'' + 25y = 12 \cos t$; \text{ solution } $y = 2 \sin 5t - 6 \cos 5t + \frac{1}{2} \cos t$
17. $y'' - y' - 2y = 0; \text{ solution } y = c_1e^{-t} + c_2e^{2t}$
18. $y'' + 2y' - 3y = 5e^{2t}; \text{ solution } y = c_1e^{-3t} + c_2e^t + e^{2t}$
19. $y'' + 6y' + 25y = 0; \text{ solution } y = e^{-3t}(c_1 \sin 4t + c_2 \cos 4t)$
20. $y'' + 8y' + 25y = 50; \text{ solution } y = e^{-3t}(c_1 \sin 3t + c_2 \cos 3t) + 2$
21. $ty'' - (t + 1)y' + y = 0, \ t > 0; \text{ solution } y = c_1e^t + c_2(t + 1)$
22. $t^2y'' + 2ty' - 2y = 5t^3, \ t > 0; \text{ solution } y = c_1t^{-2} + c_2t + \frac{1}{2}$

23–26. General solutions Two solutions of each of the following differential equations are given. If possible, give a general solution of the equation.

23. $y'' - 36y = 0; \text{ solutions } \{ e^{6t}, 5e^{-6t} \}$
24. $y'' + 5y = 0; \text{ solutions } \{ \cos \sqrt{5}t, \sin \sqrt{5}t \}$
25. $y'' + 2y' + y = 0; \text{ solutions } \{ e^{-t}, e^{3t} \}$
26. $t^2y'' + ty' - y = 0, \ t > 0; \text{ solutions } \{ t, t^{-1} \}$

27–30 Particular solutions Verify by substitution that the given functions are particular solutions of the following equations.

27. $y'' - y = 8e^{-3t}; \text{ particular solution } e^{-3t}$

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28. \(y'' + y = 3 \cos 2t;\) particular solution 2 sin \(t - \cos 2t\)
29. \(y'' - 4y' + 4y = 2e^{2t};\) particular solution \(t^2 e^{2t}\)
30. \(t^2y'' + ty' - 4y = 6t, t > 0;\) particular solution \(-2t + t^2\)

31–34. **Particular solutions are not unique** Two functions are given for each of the following differential equations. Show that both functions are particular solutions and that they differ by a solution of the homogeneous equation.

31. \(y'' - 49y = -24e^{-t};\) particular solutions \(\left\{ e^{-t}, e^t, 3e^{7t}\right\}\)
32. \(y'' + 16y = 30 \sin t;\) particular solutions \(\left\{ 2 \sin t, 2 \sin t - 8 \cos 4t\right\}\)
33. \(y'' - y' - 12y = 12e^t;\) particular solutions \(\left\{-e^t, 6e^{6t} - e^t\right\}\)
34. \(t^2y'' + 2ty' - 30y = 12r^2, t > 0;\) particular solutions \(\left\{ -\frac{t^2}{2}, 3r^3 - \frac{t^2}{2}\right\}\)

35–38. **General solutions of nonhomogeneous equations** Three solutions of the following differential equations are given. Determine which two functions are solutions of the homogeneous equation and then write the general solution of the nonhomogeneous equation.

35. \(y'' + 2y = 3e^t;\) solutions \(\left\{ \sin \sqrt{2}t, e^t, \cos \sqrt{2}t\right\}\)
36. \(y'' - 4y = 5 \cos t;\) solutions \(\left\{ 5e^{2t}, e^{-2t} - \cos t\right\}\)
37. \(y'' - 3y' + \frac{25}{4}y = 625t;\) solutions \(\left\{ e^{5t/2} \cos 2t, e^{5t/2} \sin 2t, 48 + 100t\right\}\)
38. \(t^2y'' + 2ty' - 6y = 7t^2, t > 0;\) solutions \(\left\{ -\frac{t^3}{2}, \frac{t^3}{2}\right\}\)

39–46. **Initial value problems** Solve the following initial value problems using the given general solution.

39. \(y'' + 9y = 0; y(0) = 4, y'(0) = 0;\) general solution \(y = c_1 \sin 3t + c_2 \cos 3t\)
40. \(y'' - y = 0; y(0) = 2, y'(0) = -2;\) general solution \(y = c_1 e^t + c_2 e^{-t}\)
41. \(y'' - y' - 20y = 0; y(0) = 3, y'(0) = -3;\) general solution \(y = c_1 e^{4t} + c_2 e^{-4t}\)
42. \(y'' + 4y = 5 \cos 3t; y(0) = 4, y'(0) = 2;\) general solution \(y = c_1 \sin 2t + c_2 \cos 2t - \cos 3t\)
43. \(y'' - 16y = 16t^2; y(0) = 0, y'(0) = 0;\) general solution \(y = c_1 e^{4t} + c_2 e^{-4t} - t^2 - \frac{1}{8}\)
44. \(t^2y'' + 2ty' - 2y = 0; y(1) = 3, y'(1) = 0;\) general solution \(y = c_1 t^2 + c_2 t\)
45. \(t^2y'' + ty' - 4y = 0; y(1) = 1, y'(1) = -1;\) general solution \(y = c_1 t^2 + c_2 t^2\)
46. \(y'' + 8y' + 25y = 0; y(0) = 1, y'(0) = -1;\) general solution \(y = e^{-5t}(c_1 \sin 3t + c_2 \cos 3t)\)

**Further Explorations**

47. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

a. The general solution of a second-order linear differential equation could be \(y = ce^{rt};\) where \(c\) is an arbitrary constant.

b. If \(y_3\) is a solution of a homogeneous differential equation \(y'' + py' + qy = 0\) and \(y_1\) is a particular solution of the equation \(y'' + py' + qy = f,\) then \(y_3 + cy_1\) is also a particular solution, for any constant \(c.\)

c. The functions \(\left\{ 1 - \cos^2 x, 5 \sin^2 x \right\}\) are linearly independent on the interval \([0, 2\pi]\).

d. If \(y_1\) and \(y_2\) are solutions of the equation \(y'' + y = 0,\) then \(y_1 + y_2\) is also a solution of the equation.

e. The initial value problem \(y'' + 2y = 0, y(0) = 4\) has a unique solution.

48–53. **Solution verification** Verify by substitution that the following differential equations are satisfied by the given functions. Assume that \(c_1\) and \(c_2\) are arbitrary constants.

48. \(y'' - 12y' + 36y = 0;\) solution \(y(t) = c_1 e^{6t} + c_2 e^{-6t}\)
49. \(y'' - 12y' + 36y = 2e^{6t};\) solution \(y = c_1 e^{6t} + c_2 e^{-6t} + t^2 e^{6t}\)
50. \(y'' + 4y = 8 \sin 2t;\) solution \(y = c_1 \sin 2t + c_2 \cos 2t - 2t \cos 2t\)
51. \(t^3y'' - 3ty' + 4y = 0, t > 0;\) solution \(y = c_1 t^2 + c_2 \ln t\)
52. \(t^2y'' - 3ty' + 4y = 2r^2, t > 0;\) solution \(y = c_1 r^2 + c_2 r^2 \ln t + t^2 \ln t\)
53. \(t^3y'' + ty' + \left( t^2 - \frac{1}{4}\right)y = 0, t > 0;\) solution \(y = r^{1/2}(c_1 \cos r + c_2 \sin r)\)

54. **Trigonometric solutions**

a. Verify by substitution that \(y = \sin t\) and \(y = \cos t\) are solutions of the equation \(y'' + y = 0.\)

b. Write the general solution of \(y'' + y = 0.\)

c. Verify by substitution that \(y = \sin 2t\) and \(y = \cos 2t\) are solutions of the equation \(y'' + 4y = 0.\)

d. Write the general solution of \(y'' + 4y = 0.\)

e. Based on the results of parts (a)–(d), find the general solution of the equation \(y'' + k^2 y = 0,\) where \(k\) is a nonzero real number.

55. **Hyperbolic functions** Recall that the hyperbolic sine and cosine are defined by \(\sinh t = \frac{e^t - e^{-t}}{2}\) and \(\cosh t = \frac{e^t + e^{-t}}{2}.\)

a. Verify that \(y = e^t\) and \(y = e^{-t}\) are linearly independent solutions of the equation \(y'' - y = 0.\)

b. Explain (without substituting) why \(y = \sinh t\) and \(y = \cosh t\) are linearly independent solutions of the same equation.

c. Verify by substitution that \(y = \sinh t\) and \(y = \cosh t\) are solutions of \(y'' - y = 0.\)

d. Give two different forms for the general solution of \(y'' - y = 0.\)
e. Verify that for any real number \( k, y = e^{kt} \) and \( y = e^{-kt} \) are linearly independent solutions of the equation \( y'' - k^2y = 0 \).
f. Express the general solution of \( y'' - k^2y = 0 \) in terms of \( \{ e^{kt}, e^{-kt} \} \) and \( \{ \sinh kt, \cosh kt \} \).

56–57. Higher-order equations Verify by substitution that the following equations are satisfied by the given functions.

56. \( y'' + 2y' - y - 2y = 0 \);
   solution \( y = c_1 e^{-2t} + c_2 e^{t} + e^{t} \)

57. \( y^{(4)} - 16y = 0 \); solution
   \( y = c_1 e^{-2t} + c_2 e^{2t} + c_3 \sin 2t + c_4 \cos 2t \)

58–59. Nonlinear equations

58. Find the general solution of the equation \( y'' - 2yy' = 0 \) using the following steps.
   a. Use the Chain Rule to show that \( \frac{d}{dt}(y(t)^2) = 2y(t)y'(t) \).
   b. Write the original differential equation as \( y'(t) - (y(t))^2 = 0 \).
   c. Integrate both sides of the equation in part (b) with respect to \( t \) to obtain the first-order separable equation \( y' = \pm \sqrt{2t + c_1} \), where \( c_1 \) is an arbitrary constant.
   d. Solve this equation to show that there are two families of solutions, \( y = c_2 + \frac{1}{3}(2t + c_1)^{3/2} \) and \( y = c_2 - \frac{1}{3}(2t + c_1)^{3/2} \), where \( c_2 \) is an arbitrary constant.

59. Find the general solution of the equation \( y'y' = 1 \) using the following steps.
   a. Use the Chain Rule to show that \( \frac{d}{dt}(y'(t)^2) = 2y'(t)y''(t) \).
   b. Write the original differential equation as \( (y'(t))^2 = 2 \).
   c. Integrate both sides of the equation in part (b) with respect to \( t \) to obtain the first-order equation \( y' = \pm \sqrt{2t + c_1} \), where \( c_1 \) is an arbitrary constant.
   d. Solve this equation to show that there are two families of solutions, \( y = c_2 + \frac{1}{3}(2t + c_1)^{3/2} \) and \( y = c_2 - \frac{1}{3}(2t + c_1)^{3/2} \), where \( c_2 \) is an arbitrary constant.

60–63. Not really second-order An equation of the form \( y'' = F(t, y') \) (where \( F \) does not depend on \( y \)) can be viewed as a first-order equation in \( y' \). It may be attacked in two steps: (a) Let \( v = y' \) and solve the first-order equation \( v' = F(t, v) \). (b) Having determined \( v \), solve the first order equation \( y' = v \). Use this method to find the general solution of the following equations. The methods of Sections 8.3 and 8.4 may be helpful.

60. \( y'' = 2y' \)
61. \( y'' = 3y' + 4 \)
62. \( y'' = e^{y'} \)
63. \( y'' = 2(y')^2 \)

Applications

64–67. Oscillator and circuit equations As will be shown in Section 16.4, the equation \( y'' + py' + qy = f(t) \), where \( p \) and \( q \) are constants and \( f \) is a specified function, is used to model both mechanical oscillators and electrical circuits. Depending on the values of \( p \) and \( q \), the solutions to this equation display a wide variety of behavior.

a. Verify that the following equations have the given general solution.
b. Solve the initial value problem with the given initial conditions.
c. Graph the solution to the initial value problem, for \( t \geq 0 \).

64. \( y'' + 16y = 0 \); \( y(0) = 4, y'(0) = -1 \)
   General solution \( y = c_1 \sin 4t + c_2 \cos 4t \)

65. \( y'' + 3y' + \frac{25}{4}y = 0 \); \( y(0) = 4, y'(0) = 0 \)
   General solution \( y = e^{-3/2}(c_1 \sin 2t + c_2 \cos 2t) \)

66. \( y'' + 9y = 8 \sin t; y(0) = 0, y'(0) = 2 \)
   General solution \( y = c_1 \sin 3t + c_2 \cos 3t + \sin t \)

67. \( y'' + 6y' + 25y = 20e^{-t}; y(0) = 2, y'(0) = 0 \)
   General solution \( y = e^{-3}(c_1 \sin 4t + c_2 \cos 4t) + e^{-t} \)

68. A pursuit problem Imagine a dog standing at the origin and its master standing on the positive x-axis one mile from the origin (see figure). At the same instant the dog and master begin walking. The dog walks along the positive y-axis at 1 mile per hour and the master walks at \( s > 1 \) miles per hour on a path that is always directed at the moving dog. The path followed by the master in the xy-plane is the solution of the initial value problem

\[
y''(x) = \frac{\sqrt{1 + y'(x)^2}}{s}\]

\( y(1) = 0, y'(1) = 0 \)

Solve this initial value problem using the following steps.

a. Notice that the equation is first-order in \( y' \); so let \( u = y' \), which results in the initial value problem

\[
u' = \frac{\sqrt{1 + u^2}}{sx}, u(1) = 0.
\]

b. Solve this separable equation using the fact that

\[
\frac{du}{\sqrt{1 + u^2}} = \ln \left( u + \sqrt{1 + u^2} \right) + c_1
\]

to obtain the general solution \( u + \sqrt{1 + u^2} = c_1 x^{1/2} \).

c. Use the initial condition \( u(1) = 0 \) to evaluate \( c_1 \) and show that \( u = \frac{1}{2}(x^{1/2} - x^{-1/2}) \).

d. Now recall that \( u = y' \). Solve the equation

\[
y = y' = \frac{1}{2}(x^{1/2} - x^{-1/2})
\]

by integrating both sides with respect to \( x \).
e. Use the initial condition $y(1) = 0$ to evaluate the constant of integration.

f. Conclude that the path of the master is given by
$$y = \frac{sx}{2} - \frac{x^{-1/3}}{s + 1} + \frac{s}{s^2 - 1}.$$ 

g. Graph the pursuit paths for $s = 1.1, 1.3, 1.5, 2.0$. Explain the dependence on $s$ that you observe.

### Additional Exercises

#### 69. Conservation of Energy

In some cases, Newton’s second law can be written $m\ddot{x}(t) = F(x)$, where the force $F$ depends only on the position $x$, and there is a function $\varphi$ (called a potential) such that $\varphi’(x) = -F(x)$. Systems with this property obey an energy conservation law.

a. Multiply the equation of motion by $x’(t)$ and show that 
$$\frac{d}{dt} \left( \frac{1}{2}m(x’(t))^2 + \varphi(x) \right) = 0.$$  

b. Define the energy of the system to be $E(t) = \frac{1}{2}mv^2 + \varphi$ (the sum of kinetic and potential energy) and show that $E(t)$ is constant in time.

#### 70. Reduction of Order

Suppose you are solving a second-order linear homogeneous differential equation and you have found one solution. A method called reduction of order allows you to find the second (linearly independent) solution (up to evaluating integrals). Consider the differential equation $y'' - \frac{1}{t}y’ + \frac{1}{t^2}y = 0$, for $t > 0$.

a. Verify that $y_1 = t$ is a solution. Assume the second homogeneous solution is $y_2$ and it has the form $y_2(t) = v(t)y_1(t) = vt(t)$, where $v$ is a function to be determined.

b. Substitute $y_2$ into the differential equation and simplify the resulting equation to show that $v$ satisfies the equation $v'' = -\frac{v’}{t}$.

c. Note that this equation is first order in $v’$; so let $w = v’$ to obtain the first-order equation $w’ = -\frac{w}{t}$.

d. Solve this separable equation and show that $w = \frac{c_1}{t}$.

e. Now solve the equation $v’ = w = \frac{c_1}{t}$ to find $v$.

f. Finally, recall that $y_2(t) = vt(t)$ and conclude that the second solution is $y_2(t) = c_1t \ln t$.

#### Quick Check Answers

1. First order, linear, nonhomogeneous; second order, linear, homogeneous
2. The first, third, and fourth pairs are linearly independent. The second pair is linearly dependent.
3. Yes.
4. $c_1 = 0, c_2 = 1.$

### 16.2 Linear Homogeneous Equations

Up until now, you have been given a function and asked to verify by substitution that it satisfies a particular differential equation. Now it’s time to carry out the actual solution process. We begin with the case of constant-coefficient homogeneous equations of the form
$$y''(t) + py'(t) + qy(t) = 0,$$

where $p$ and $q$ are constants.

We solve this equation by making the following observation: A solution of this equation is a function $y$ whose derivatives $y'$ and $y''$ are constant multiples of $y$ itself, for all $t$. The only functions with this property have the form $y = e^{rt}$, where $r$ is a constant.

This observation suggests using a trial solution of the form $y = e^{rt}$, where $r$ must be determined. We substitute the trial solution into the equation and carry out the following calculation.

$$\left( e^{rt} \right)^n + p(e^{rt})’ + qe^{rt} = 0$$  Substitute.

$$r^n e^{rt} + pre^{rt} + qe^{rt} = 0$$  Differentiate.

$$e^{rt}(r^2 + pr + q) = 0$$  Factor $e^{rt}$.

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Recall that our aim is to find values of $r$ that satisfy this equation for all $t$. We may cancel the factor $e^{rt}$ because it is nonzero for all $t$. What remains after canceling $e^{rt}$ is a quadratic (second-degree) equation

$$r^2 + pr + q = 0,$$

which can be solved for the unknown $r$. The polynomial $r^2 + pr + q$ is called the characteristic polynomial (or auxiliary polynomial) for the differential equation.

It is important to see what the roots of the characteristic polynomial look like. Using the quadratic formula, they are

$$r_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad r_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Recall that these three cases arise.

- If $p^2 - 4q > 0$, then the roots are real with $r_1 \neq r_2$, and they are expressed exactly as in expression (1).
- If $p^2 - 4q = 0$, then the polynomial has the repeated root $r_1 = \frac{-p}{2}$.
- If $p^2 - 4q < 0$, then polynomial has a pair of complex roots

$$r_1 = \frac{-p + i\sqrt{4q - p^2}}{2} \quad \text{and} \quad r_2 = \frac{-p - i\sqrt{4q - p^2}}{2}.$$

These three cases produce different types of solutions to the differential equation, and we must examine them individually.

**Case 1: Real Distinct Roots of the Characteristic Polynomial**

Suppose that $p^2 - 4q > 0$ and the roots of the characteristic polynomial are real numbers $r_1$ and $r_2$, with $r_1 \neq r_2$. We assumed that solutions of the differential equation have the form $y = e^{rt}$. Therefore, we have found two solutions, $y_1 = e^{r_1t}$ and $y_2 = e^{r_2t}$, which are linearly independent because $r_1 \neq r_2$. Using what we learned in Section 16.1, the general solution of the differential equation consists of linear combinations of these two functions:

$$y = c_1y_1 + c_2y_2 = c_1e^{r_1t} + c_2e^{r_2t}.$$

**EXAMPLE 1**  **General solution with real distinct roots**  
Find the general solution of the differential equation

$$y'' - 2y' - 4y = 0.$$

**SOLUTION**  
We form the characteristic polynomial directly from the differential equation; the equation that must be solved is

$$r^2 - 2r - 4 = 0.$$

Using the quadratic formula, the roots are found to be

$$r_1 = 1 + \sqrt{5} \quad \text{and} \quad r_2 = 1 - \sqrt{5}.$$

Therefore, the general solution is

$$y = c_1e^{(1+\sqrt{5})t} + c_2e^{(1-\sqrt{5})t},$$

where $c_1$ and $c_2$ are arbitrary constants.

**Related Exercises 9–14**

**EXAMPLE 2**  **Initial value problem with real distinct roots**  
Solve the initial value problem

$$y'' + y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = -5.$$
**SOLUTION** To find the general solution, we find the roots of the characteristic polynomial, which satisfy

\[ r^2 + r - 6 = 0. \]

Factoring the polynomial or using the quadratic formula, the roots are \( r_1 = 2 \) and \( r_2 = -3 \). Therefore, the general solution is

\[ y = c_1 e^{2t} + c_2 e^{-3t}. \]

The arbitrary constants \( c_1 \) and \( c_2 \) are now determined using the initial conditions. Noting that \( y'(t) = 2c_1 e^{2t} - 3c_2 e^{-3t} \), the initial conditions imply that

\[ y(0) = c_1 e^{2*0} + c_2 e^{-3*0} = c_1 + c_2 = 0 \]
\[ y'(0) = 2c_1 e^{2*0} - 3c_2 e^{-3*0} = 2c_1 - 3c_2 = -5. \]

Solving these two equations gives the constants \( c_1 = -1 \) and \( c_2 = 1 \). The solution of the initial value problem now follows; it is

\[ y = -e^{2t} + e^{-3t}. \]

Figure 16.4 shows that the solution to the initial value problem (in red) is one of infinitely many functions of the general solution. It is the only function that satisfies the initial conditions.

![Figure 16.4](https://example.com/fig16_4.png)

**FIGURE 16.4**

Related Exercises 15–20

**Case 2: Real Repeated Roots of the Characteristic Polynomial**

We now assume that \( p^2 - 4q = 0 \), which means the only root of the characteristic polynomial is

\[ r = -p + \sqrt{p^2 - 4q} = \frac{-p}{2} \]

This one root produces the solution \( y_1 = c_1 e^{r_1 t} \), but where do we find a second (linearly independent) solution? It may be found by making an ingenious assumption followed by a short calculation.

Because the first solution has the form \( y_1 = c_1 e^{r_1 t} \), where \( c_1 \) is a constant, we look for a second solution that has the form \( y_2 = v(t) e^{r_1 t} \), where \( v(t) \) is not a constant, but a
function of \( t \) that must be determined. In the spirit of a trial solution, we substitute \( y_2 \) into the differential equation and see where it takes us.

By the Product Rule

\[
y_2'(t) = v'(t)e^{rt} + v(t)r_1e^{rt} \quad \text{and} \quad y_2''(t) = v''(t)e^{rt} + 2v'(t)r_1e^{rt} + v(t)r_1^2e^{rt}.
\]

We now substitute \( y_2 \) into the differential equation \( y'' + pty' + qy = 0 \):

\[
\begin{align*}
v''e^{rt} + 2v'r_1e^{rt} + vr_1^2e^{rt} + p(v'e^{rt}) + v'r_1e^{rt} + qve^{rt} &= 0 \\
= e^{rt}(v'' + (2r_1 + p)v' + vr_1^2 + pr_1 + q) &= 0 \\
= e^{rt}v'' &= 0.
\end{align*}
\]

We used the fact that \( 2r_1 + p = 0 \) because \( r_1 = -\frac{p}{2} \). In addition, \( r_1 \) is a root of the characteristic polynomial, which implies that \( r_1^2 + pr_1 + q = 0 \). After making these simplifications, we are left with the equation \( e^{rt}v''(t) = 0 \). Because \( e^{rt} \) is nonzero for all \( t \), we cancel this factor, leaving an equation for the unknown function \( v \); it is simply \( v''(t) = 0 \).

We solve this equation by integrating once to give \( v'(t) = c_1 \), and then again to give \( v(t) = c_1t + c_2 \), where \( c_1 \) and \( c_2 \) are arbitrary constants. Remember that this calculation begins by assuming that the second homogeneous solution has the form \( y_2 = v(t)e^{rt} \). Now that we have found \( v \), we can write

\[
y_2 = v(t)e^{rt} = (c_1t + c_2)e^{rt} = c_1te^{rt} + c_2e^{rt}
\]

This calculation has produced the first solution \( y_1 = e^{rt} \), as well as the second solution that we sought, \( y_2 = te^{rt} \). So the mystery is solved. The two linearly independent solutions are \( \{e^{rt}, te^{rt}\} \), and the general solution in the repeated root case is

\[
y = c_1e^{rt} + c_2te^{rt}.
\]

**EXAMPLE 3 Initial value problem with repeated roots** Solve the initial value problem

\[
y'' + 4y' + 4y = 0, \quad y(0) = 8, \quad y'(0) = 4.
\]

**SOLUTION** Solving the equation

\[
r^2 + 4r + 4 = (r + 2)^2 = 0,
\]

the characteristic polynomial has the single repeated root \( r_1 = -2 \). Therefore, the general solution of the differential equation is

\[
y = c_1e^{-2t} + c_2te^{-2t}.
\]

We appeal to the initial conditions to evaluate the constants in the general solution. In this case,

\[
y'(t) = -2c_1e^{-2t} + c_2(e^{-2t} - 2te^{-2t}) = e^{-2t}(-2c_1 + c_2 - 2c_2).
\]

The initial conditions imply that

\[
y(0) = c_1e^{-2\cdot0} + c_2\cdot0\cdot e^{-2\cdot0} = c_1 = 8
\]

\[
y'(0) = e^{-2\cdot0}(-2c_1 + c_2 - 2\cdot0\cdot c_2) = -2c_1 + c_2 = 4.
\]
Solving these two equations gives the solutions $c_1 = 8$ and $c_2 = 20$. The solution of the initial value problem is

$$y = 8e^{-2t} + 20te^{-2t}.$$  

The behavior of this solution is worth investigating because solutions of this form arise in practice. Figure 16.5 shows several functions of the general solution along with the function that satisfies the initial value problem (in red). Of particular importance is the fact that for all these solutions, $\lim_{t \to \infty} y(t) = 0$. In general, when $a > 0$, we have $\lim_{t \to \infty} te^{-at} = 0$ and $\lim_{t \to \infty} e^{at} = \infty$.

![Figure 16.5](image-url)

**Case 3: Complex Roots of the Characteristic Polynomial**

The third case arises when $p^2 - 4q < 0$, which implies that the roots of the characteristic polynomial occur in complex conjugate pairs. The roots are

$$r_1 = \frac{-p + i\sqrt{4q - p^2}}{2} \quad \text{and} \quad r_2 = \frac{-p - i\sqrt{4q - p^2}}{2},$$

which we abbreviate as $r_1 = a + ib$ and $r_2 = a - ib$, where $a = \frac{-p}{2}$ and $b = \frac{\sqrt{4q - p^2}}{2}$ are real numbers. It is easy to write the general solution of the differential equation as

$$y = c_1e^{r_1t} + c_2e^{r_2t} = c_1e^{(a+ib)t} + c_2e^{(a-ib)t},$$

but what does it mean? We expect a real-valued solution to a differential equation with real coefficients. A bit of work is required to express this solution with real-valued functions. Using properties of exponential functions, we first factor $e^{at}$ and write

$$y = e^{at}(c_1e^{ibt} + c_2e^{-ibt}).$$

Written in this form, we see that two solutions of the differential equation are $e^{at}e^{ibt}$ and $e^{at}e^{-ibt}$. Now recall that linear combinations of solutions are also solutions. We use the facts that

$$\cos bt = \frac{e^{ibt} + e^{-ibt}}{2} \quad \text{and} \quad \sin bt = \frac{e^{ibt} - e^{-ibt}}{2i}$$

to form the following linear combinations:

$$\frac{1}{2}e^{at}e^{ibt} + \frac{1}{2}e^{at}e^{-ibt} = e^{at}\cdot \frac{e^{ibt} + e^{-ibt}}{2} = e^{at}\cos bt$$

$$\frac{1}{2i}e^{at}e^{ibt} - \frac{1}{2i}e^{at}e^{-ibt} = e^{at}\cdot \frac{e^{ibt} - e^{-ibt}}{2i} = e^{at}\sin bt.$$
Now we have two real-valued, linearly independent solutions: $e^{at} \cos bt$ and $e^{at} \sin bt$. Therefore, in the case of complex roots, the general solution is

$$y = c_1 e^{at} \cos bt + c_2 e^{at} \sin bt,$$

where $a = -\frac{p}{2}$ and $b = \frac{\sqrt{4q - p^2}}{2}$. Recall that the roots of the characteristic polynomial are $a \pm ib$. Therefore, the real part of each root is $a$, which determines the rate of exponential growth or decay of the solution. The imaginary part of each root is $b$, which determines the period of oscillation of the solution; we see that the period is $2\pi/b$.

**EXAMPLE 4** Initial value problem with complex roots Solve the initial value problem

$$y'' + 16y = 0, \quad y(0) = -2, \quad y'(0) = 6.$$

**SOLUTION** The roots of the characteristic polynomial satisfy $r^2 + 16 = 0$; in this case, we have the pure imaginary roots $r_1 = 4i$ and $r_2 = -4i$. Therefore, the general solution $y = c_1 e^{at} \cos bt + c_2 e^{at} \sin bt$ with $a = 0$ and $b = 4$ becomes

$$y = c_1 \cos 4t + c_2 \sin 4t.$$

Before using the initial conditions, we compute $y'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$. The initial conditions imply that

$$y(0) = c_1 \cos (4 \cdot 0) + c_2 \sin (4 \cdot 0) = c_1 = -2,$$

$$y'(0) = -4c_1 \sin (4 \cdot 0) + 4c_2 \cos (4 \cdot 0) = 4c_2 = 6.$$  

We conclude that $c_1 = -2$ and $c_2 = \frac{3}{2}$, making the solution of the initial value problem

$$y = -2 \cos 4t + \frac{3}{2} \sin 4t.$$

The solution is shown in **Figure 16.6** (in red), along with several other functions of the general solution. When the roots of the characteristic polynomial are pure imaginary numbers, as in this case, the solution is oscillatory with no growth or attenuation of the solution. In this case, with $b = 4$, the period of the solution is $2\pi/4 = \pi/2$.  

**FIGURE 16.6** Related Exercises 27–32
**EXAMPLE 5** Initial value problem with complex roots. Solve the initial value problem

\[ y'' + y' + \frac{5}{4}y = 0, \quad y(0) = 2, \quad y'(0) = 2. \]

**SOLUTION** Using the quadratic formula, the characteristic polynomial \( r^2 + r + \frac{5}{4} \) has roots \( r_1 = -\frac{1}{2} + i \) and \( r_2 = -\frac{1}{2} - i \). Identifying \( a = -\frac{1}{2} \) and \( b = 1 \), the general solution is

\[ y = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t. \]

Before using the initial conditions, we compute

\[ y'(t) = c_1 \left(-\frac{1}{2}e^{-t/2} \cos t - e^{-t/2} \sin t\right) + c_2 \left(-\frac{1}{2}e^{-t/2} \sin t + e^{-t/2} \cos t\right). \]

The initial conditions imply that

\[ y(0) = c_1 = 2 \]
\[ y'(0) = -\frac{1}{2}c_1 + c_2 = 2. \]

These conditions are satisfied provided \( c_1 = 2 \) and \( c_2 = 3 \). Therefore, the solution to the initial value problem is

\[ y = 2e^{-t/2} \cos t + 3e^{-t/2} \sin t. \]

The solution is a wave with an attenuated amplitude (Figure 16.7). The damped wave fits nicely within an envelope formed by functions of the form \( y = \pm Ae^{-t/2} \) (dashed curves).

**Figure 16.8**

Figure 16.8 gives a graphical interpretation in the \( pq \)-plane of the three cases that arise in solving the equation \( y'' + py' + qy = 0 \). We see that the parabola \( q = \frac{p^2}{4} \) divides the plane into two regions. Values of \((p, q)\) above the parabola correspond to equations whose characteristic polynomials have complex roots, whereas those values below the parabola correspond to the case of real distinct roots. The parabola itself represents the case of repeated real roots. Table 16.1 also summarizes the three cases.

---

**FIGURE 16.7**

Related Exercises 27–32
Table 16.1 Cases for the equation \( y'' + py' + qy = 0 \)

<table>
<thead>
<tr>
<th>Roots ( r )</th>
<th>General solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p^2 - 4q &gt; 0 )</td>
<td>( r_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2} )</td>
</tr>
<tr>
<td>( p^2 - 4q = 0 )</td>
<td>( r_1 = r_2 = \frac{-p}{2} )</td>
</tr>
<tr>
<td>( p^2 - 4q &lt; 0 )</td>
<td>( r_{1,2} = a \pm ib, a = \frac{-p}{2}, b = \frac{\sqrt{4q - p^2}}{2} )</td>
</tr>
</tbody>
</table>

Amplitude-Phase Form

We pause here to mention two techniques that appear in upcoming work. The use of the amplitude-phase form of a solution may be familiar, but it’s worth reviewing. A function of the form \( y = c_1 \sin \omega t + c_2 \cos \omega t \) (which arises in solutions in Case 3 above) is difficult to visualize. However, functions of this form may always be expressed in the form \( y = A \sin (\omega t + \varphi) \) or \( y = A \cos (\omega t + \varphi) \). If we choose \( y = A \sin (\omega t + \varphi) \), the relationships among the amplitude \( A \), the phase \( \varphi \), and \( c_1 \) and \( c_2 \) are

\[
A = \sqrt{c_1^2 + c_2^2} \quad \text{and} \quad \tan \varphi = \frac{c_2}{c_1}.
\]

(See Exercise 40; Exercise 41 gives similar expressions for \( A \cos (\omega t + \varphi) \).) The function \( y = A \sin (\omega t + \varphi) \) is a shifted sine function with constant amplitude \( A \) and frequency \( \omega \). For example, consider the function \( y = -2 \sin 3t + 2 \cos 3t \). Letting \( c_1 = -2 \) and \( c_2 = 2 \), we have

\[
A = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2} \quad \text{and} \quad \tan \varphi = \frac{2}{-2}.
\]

which implies that \( \varphi = \frac{3\pi}{4} \). Therefore, the function can also be written as

\[
y = 2\sqrt{2} \sin \left(3t + \frac{3\pi}{4}\right) = 2\sqrt{2} \sin 3\left(t + \frac{\pi}{4}\right).
\]

The function is now seen to be a sine wave with amplitude \( 2\sqrt{2} \) and period \( \frac{2\pi}{3} \), shifted \( \frac{\pi}{4} \) units to the left (Figure 16.9).

![Figure 16.9](image-url)

\[
y = -2 \sin 3t + 2 \cos 3t = 2\sqrt{2} \sin (3t + 3\pi/4)
\]
The Phase Plane

In the remainder of this chapter, we occasionally use the *phase plane* to display solutions of differential equations. Rather than graph the solution $y$ as a function of $t$, we instead make a parametric plot of $y$ and $y'$. The phase plane reveals features of the solution that may not be apparent in the usual time-dependent graph.

Consider the periodic function $y = \sin t - 2 \cos t$, whose graph is shown in Figure 16.10a. In the phase-plane graph of the function (Figure 16.10b), the parameter $t$ does not appear explicitly. However, the curve has an orientation (indicated by the arrow) that shows the direction of increasing $t$. Any point on the curve corresponds to at least one solution value; for example, the point $(y(0), y'(0))$ (shown on the curve) is also associated with $t = 2\pi, 4\pi, \ldots$. The fact that the curve is closed reflects the fact that the function is periodic.

![Figure 16.10](image1)

**FIGURE 16.10**

In contrast, consider the function $y = e^{-t/4}(\sin t - 2 \cos t)$, whose graph is shown in Figure 16.11a. The phase-plane plot (Figure 16.11b) is an inward spiral that gives a distinctive picture of the decaying amplitude of the function.

![Figure 16.11](image2)

**FIGURE 16.11**
The Cauchy-Euler Equation

We close this section with a brief look at a second-order linear \textit{variable-coefficient} equation that can also be solved using roots of polynomials. The \textbf{Cauchy-Euler} (or \textit{equi-dimensional}) \textbf{equation} has the form

\[ t^2y''(t) + at'(t) + by(t) = 0, \]

where \( a \) and \( b \) are constants and \( t > 0 \). The defining feature of this equation is that in each term the power of \( t \) matches the order of the derivative. Assuming \( t > 0 \), both sides of the equation may be divided by \( t^2 \) to produce the equation

\[ y''(t) + \frac{a}{t}y'(t) + \frac{b}{t^2}y(t) = 0. \]

We see that the coefficients of \( y' \) and \( y \) are not continuous on any interval containing \( t = 0 \). For this reason, initial value problems associated with this equation are posed on intervals that do not include the origin.

The equation is solved using a trial solution of the form \( y = t^p \), where the exponent \( p \) must be determined. Substituting the trial solution into the differential equation, we find that

\begin{align*}
    t^2(p^p)'' + ap(p')' + b(p^p) &= 0 & \text{Substitute trial solution.} \\
    p(p - 1)t^p + apt^p + btp^p &= 0 & \text{Differentiate.} \\
    t^p(p(p - 1) + ap + b) &= 0 & \text{Collect terms.} \\
    t^p(p^2 + (a - 1)p + b) &= 0. & \text{Simplify.}
\end{align*}

If we assume that \( t > 0 \), then \( t^p > 0 \) and we may divide through the equation by \( t^p \). Doing so leaves a polynomial equation to be solved for the unknown \( p \). When the quadratic equation \( p^2 + (a - 1)p + b = 0 \) is solved, we again have three cases. If the roots are real and distinct, call them \( p_1 \) and \( p_2 \) with \( p_1 \neq p_2 \), then we have two linearly independent solutions \( \{ t^{p_1}; t^{p_2} \} \). The general solution of the differential equation is

\[ y = c_1t^{p_1} + c_2t^{p_2}. \]

The cases in which the roots are real and repeated, and in which the roots are complex, are examined in Exercises 52–59 and 62–64.

\textbf{EXAMPLE 6} \textbf{Cauchy-Euler initial value problem} \hspace{1em} Solve the initial value problem

\[ t^2y'' + 2ty' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 3. \]

\textbf{SOLUTION} \hspace{1em} Substituting the trial solution \( y = t^p \) into the differential equation produces the polynomial

\[ p^2 + p - 2 = (p - 1)(p + 2) = 0. \]

The roots are \( p_1 = 1 \) and \( p_2 = -2 \), which gives the general solution

\[ y = c_1t + c_2t^{-2}. \]

To impose the initial conditions, we must compute

\[ y'(t) = c_1 - 2c_2t^{-3}. \]

The initial conditions now imply that

\begin{align*}
    y(1) &= c_1 + c_2 = 0 \\
    y'(1) &= c_1 - 2c_2 = 3.
\end{align*}

The solution of this set of equations is \( c_1 = 1 \) and \( c_2 = -1 \). Therefore, the solution of the initial value problem is

\[ y = t - t^{-2}. \]

Several functions of the general solution along with the solution of the initial value problem (in red) are shown in \textbf{Figure 16.12}. Related Exercises 33–38
SECTION 16.2 EXERCISES

Review Questions

1. Give the trial solution used to solve linear constant-coefficient homogeneous differential equations.
2. What is the characteristic polynomial associated with the equation \( y'' - 3y' + 10 = 0 \)?
3. Give the three cases that arise when finding the roots of the characteristic polynomial.
4. What is the form of the general solution of a second-order constant-coefficient equation when the characteristic polynomial has two distinct real roots?
5. What is the form of the general solution of a second-order constant-coefficient equation when the characteristic polynomial has repeated real roots?
6. What is the form of the general solution of a second-order constant-coefficient equation when the characteristic polynomial has complex roots?
7. The characteristic polynomial for a second-order equation has roots \(-2 \pm 3i\). Give the real form of the general solution.
8. Give the trial solution used to solve a second-order Cauchy-Euler equation.

Basic Skills

9–14. General solutions with distinct real roots Find the general solution of the following differential equations.

9. \( y'' - 25y = 0 \)
10. \( y'' - 2y' - 15y = 0 \)
11. \( y'' - 3y' = 0 \)
12. \( y'' - y' - \frac{3}{4}y = 0 \)
13. \( 2y'' + 6y' - 20y = 0 \)
14. \( y'' - \frac{5}{2}y' + y = 0 \)

15–20. Initial value problems with distinct real roots Find the general solution of the following differential equations. Then solve the given initial value problem.

15. \( y'' - 36y = 0; y(0) = 3, y'(0) = 0 \)
16. \( y'' - 6y' = 0; y(0) = -1, y'(0) = 2 \)
17. \( y'' - 3y' - 18y = 0; y(0) = 0, y'(0) = 4 \)
18. \( y'' + 8y' + 15y = 0; y(0) = 2, y'(0) = 4 \)
19. \( y'' - 2y' - \frac{5}{4}y = 0; y(0) = 3, y'(0) = 0 \)
20. \( y'' - 10y' + 21y = 0; y(0) = -3, y'(0) = -1 \)

21–26. Initial value problems with repeated real roots Find the general solution of the following differential equations. Then solve the given initial value problem.

21. \( y'' - 2y' + y = 0; y(0) = 4, y'(0) = 0 \)
22. \( y'' + 6y' + 9y = 0; y(0) = 0, y'(0) = -1 \)
23. \( y'' - y' + \frac{1}{4}y = 0; y(0) = 1, y'(0) = 2 \)
24. \( y'' - 4\sqrt{2}y' + 8y = 0; y(0) = 1, y'(0) = 0 \)
25. \( y'' + 4y' + 4y = 0; y(0) = 1, y'(0) = 0 \)
26. \( y'' + 3y' + \frac{9}{4}y = 0; y(0) = 0, y'(0) = 3 \)

27–32. Initial value problems with complex roots Find the general solution of the following differential equations. Then solve the given initial value problem.

27. \( y'' + 9y = 0; y(0) = 8, y'(0) = -8 \)
28. \( y'' + 6y' + 25y = 0; y(0) = 4, y'(0) = 0 \)
29. \( y'' - 2y' + 5y = 0; y(0) = 1, y'(0) = -1 \)
30. \( y'' + 4y' + 5y = 0; y(0) = 2, y'(0) = -2 \)
31. \( y'' + 6y' + 10y = 0; y(0) = 0, y'(0) = 6 \)
32. \( y'' - y' + \frac{1}{2}y = 0; y(0) = 3, y'(0) = -2 \)

33–38. Initial value problems with Cauchy-Euler equations Find the general solution of the following differential equations, for \( t \geq 1 \). Then solve the given initial value problem.

33. \( t^2y'' + ty' - y = 0; y(1) = 2, y'(1) = 0 \)
34. \( t^2y'' + 2ty' - 12y = 0; y(1) = 0, y'(1) = 6 \)
35. \( t^2y'' - ty' - 15y = 0; y(1) = 6, y'(1) = -1 \)
36. \( t^2y'' + 4ty' - 4y = 0; y(1) = 5, y'(1) = -3 \)
37. \( t^2y'' + 6ty' + 6y = 0; y(1) = 0, y'(1) = -4 \)
38. \( t^2y'' + ty' - 2y = 0; y(1) = 8, y'(1) = -12 \)

Further Explorations

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. To solve the equation \( y'' + ty' + 4y = 0 \) you should use the trial solution \( y = e^{rt} \).

b. The equation \( y'' + ty' + 4t^2y = 0 \) is a Cauchy-Euler equation.

c. A second-order differential equation with constant real coefficients has a characteristic polynomial with roots \( 2 + 3i \) and \(-2 + 3i\).

40. The general solution of a second-order homogeneous differential equation with constant real coefficients could be \( y = c_1 \cos (2t) + c_2 \sin (2t) \).

e. The general solution of a second-order homogeneous differential equation with constant real coefficients could be \( y = c_1 \cos (2t) + c_2 \sin (2t) \).
40. **Amplitude-phase form** The goal is to express the function
\[ y = c_1 \sin \omega t + c_2 \cos \omega t \]
in the form \( y = A \sin (\omega t + \varphi) \), where \( c_1 \) and \( c_2 \) are known, and \( A \) and \( \varphi \) must be determined.

a. Use the identity \( \sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \) to expand \( y = A \sin (\omega t + \varphi) \).

b. Equate the result of part (a) to \( y = c_1 \sin \omega t + c_2 \cos \omega t \), and match coefficients of \( \sin \omega t \) and \( \cos \omega t \) to conclude that
   \[ c_1 = A \cos \varphi \]
   and \( c_2 = A \sin \varphi \).

c. Solve for \( A \) and \( \varphi \) to conclude that \( A = \sqrt{c_1^2 + c_2^2} \) and
   \[ \tan \varphi = \frac{c_2}{c_1}. \]

41. **Amplitude-phase form** The function \( y = c_1 \sin \omega t + c_2 \cos \omega t \) can also be expressed in the form \( y = A \cos (\omega t + \varphi) \), where \( c_1 \) and \( c_2 \) are known, and \( A \) and \( \varphi \) must be determined. Use the procedure in Exercise 40 with the identity
   \[ \sin (\alpha + \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \]
to conclude that
   \[ A = \sqrt{c_1^2 + c_2^2} \]
   and \( \tan \varphi = -\frac{c_1}{c_2}. \)

42–45. **Converting to amplitude-phase form** Express the following functions in the form \( y = A \sin (\omega t + \varphi) \). Check your work by graphing both forms of the function.

42. \( y = 2 \sin 3t - 2 \cos 3t \)

43. \( y = -3 \sin 4t + 3 \cos 4t \)

44. \( y = \sqrt{3} \sin t + \cos t \)

45. \( y = -\sin 2t + \sqrt{3} \cos 2t \)

46–51. **Higher-order equations** Higher-order equations with constant coefficients can also be solved using the trial solution \( y = e^{\alpha t} \) and finding roots of a characteristic polynomial. Find the general solution of the following equations.

46. \( y'' - 4y' = 0 \)

47. \( y'' - y' - 6y = 0 \)

48. \( y'' + y' = 0 \)

49. \( y'' - 6y' + 8y = 0 \)

50. \( y^{(4)} - 5y'' + 4y = 0 \)

51. \( y^{(4)} + 5y'' + 4y = 0 \)

52–55. **Cauchy-Euler equation with repeated roots** It can be shown (Exercise 62) that when the polynomial associated with a second-order Cauchy-Euler equation has the repeated root \( r = r_1 \), the second linearly independent solution is \( y = t^{r_1} \ln t \), for \( t > 0 \). Find the general solution of the following equations.

52. \( t^2y'' - ty' + y = 0 \)

53. \( t^2y'' + 3ty' + y = 0 \)

54. \( t^2y'' - 3ty' + 4y = 0 \)

55. \( t^2y'' + 7ty' + 9y = 0 \)

56–59. **Cauchy-Euler equation with complex roots** It can be shown (Exercise 64) that when the polynomial associated with a second-order Cauchy-Euler equation has complex roots \( r = \alpha \pm \beta \), the linearly independent solutions are \( r^* \cos (\beta \ln t) \), \( r^* \sin (\beta \ln t) \), for \( t > 0 \). Find the general solution of the following equations.

56. \( t^2y'' + ty' + y = 0 \)

57. \( t^2y'' + 7ty' + 25y = 0 \)

58. \( t^2y'' - ty' + 5y = 0 \)

59. \( t^2y'' + \frac{1}{2}y = 0 \)

46. **Applications**
   Section 16.4 is devoted to applications of second-order equations.

60. **Oscillators and circuits** As we show in Section 16.4, the equation \( x'' + px' + qx = 0 \) is used to model mechanical oscillators and electrical circuits in the absence of external forces (often called free oscillations). Consider this equation in the case of complex roots \( (p^2 - 4q < 0) \), in which case the general solution has the form \( e^{\alpha t}(c_1 \cos bt + c_2 \sin bt) \).

a. In the general solution, let \( a = -\frac{1}{2}, c_1 = 2 \), and \( c_2 = 0 \). Graph the solution on the interval \( 0 \leq t \leq 2\pi \), with \( b = 1 \). 2, 3, 4. What is the time interval between successive maxima of the oscillation?

b. For each function in part (a), what is the frequency of the oscillation (i) measured in cycles per unit of time and (ii) measured in units of cycles per \( 2\pi \) units of time?

c. In the general solution, let \( b = 2 \), \( c_1 = 2 \), and \( c_2 = 0 \). Graph the solution on the interval \( 0 \leq t \leq 2\pi \), with \( a = -0.1 \), \(-0.5 \), \(-1 \), \(-1.5 \), \(-2 \). In each case, how long does it take for the amplitude of the oscillation to decay to \( 1/3 \) of its initial value?

d. Graph the solution \( y = e^{\sqrt{2}}(2 \cos 2t - \sin 2t) \). What is the time interval between successive maxima of the oscillation? Roughly how long does it take for the amplitude of the oscillation to decay to \( 1/3 \) of its initial value?

61. **Buckling column** A model of the buckling of an elastic column involves the fourth-order equation \( y^{(4)}(x) + k^2y''(x) = 0 \), where \( k \) is a positive real number. Find the general solution of this equation.

62. **Cauchy-Euler equation with repeated roots** One of several ways to find the second linearly independent solution of a Cauchy-Euler equation
   \[ t^2y''(t) + aty'(t) + by(t) = 0 \]
in the case of a repeated real root is to change variables.

a. What is the polynomial associated with this equation?

b. Show that if we let \( t = e^x \) (or \( x = \ln t \)), then this equation becomes the constant coefficient equation
   \[ y''(x) + (a - 1)y'(x) + by(x) = 0. \]

c. What is the characteristic polynomial for the equation in part (b)? Conclude that if the polynomial in part (a) has a repeated root, then the characteristic polynomial also has a repeated root.

d. Write the general solution of the equation in part (b) in the case of a repeated root.

e. Express the solution in part (d) in terms of the original variable \( t \) to show that the second linearly independent solution of the Cauchy-Euler equation is \( y = t^{(1-n)/2} \ln t \).
63. Cauchy-Euler equation with repeated roots again Here is another instructive calculation that leads to the second linearly independent solution of a Cauchy-Euler equation in the case of a repeated root. Assume the equation has the form \( t^2 y'' + a t y' + by = 0 \), for \( t > 0 \), and that the corresponding polynomial has a repeated root.

a. Show that the repeated root is \( p = \frac{1 - a}{2} \), which implies that \( 2p + a = 1 \).

b. Assume the second solution has the form \( y(t) = t^p v(t) \), where the function \( v \) must be determined. Substitute the solution in this form into the equation. After simplification and using part (a), show that \( v \) satisfies the equation \( t^2 v'' + p(t)v' = 0 \).

c. This equation is first order in \( v' \), so let \( w = v' \) and solve for \( w \) to find that \( w = \frac{c_1}{t} \) where \( c_1 \) is an arbitrary constant of integration.

d. Now solve the equation \( v' = w = \frac{c_1}{t} \) for \( v \) and show that \( y(t) = t^p v(t) \) is the complete general solution (with two linearly independent solutions).

64. Cauchy-Euler equation with complex roots To find the general solution of a Cauchy-Euler equation (2) \( t^2 y'' + a t y' + by = 0 \), for \( t > 0 \) in the case of complex roots, we change variables as in Exercise 62.

a. Show that with \( t = e^x \) (or \( x = \ln t \)), the original differential equation can be written \( y''(x) + (a - 1)y'(x) + by(x) = 0 \).

b. Solve this constant-coefficient equation in the case of complex roots. Then express the solution in terms of \( t \) to show that the general solution of the Cauchy-Euler equation is \( y = c_1 t^\alpha \cos (\beta \ln t) + c_2 t^\alpha \sin (\beta \ln t) \), where \( \alpha = \frac{1 - a}{2} \) and \( \beta = \frac{1}{2} \sqrt{4b - (a - 1)^2} \).

65. Finding the second solution Consider the constant-coefficient equation \( y'' + py' + qy = 0 \) in the case that the characteristic polynomial has two distinct real roots \( r_1 \) and \( r_2 \). The general solution is \( y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \).

a. Explain why \( u = \frac{e^{r_1 t} - e^{r_2 t}}{r_1 - r_2} \) is a solution of the equation.

b. Now consider \( t \) and \( r_2 \) fixed and let \( r_1 \to r_2 \) in order to investigate what happens as the distinct roots become a repeated root. Evaluate \( \lim u \) and identify both linearly independent solutions in the repeated root case.

66. Finding the second solution again Consider the constant-coefficient equation \( y'' + py' + qy = 0 \) in the case that the characteristic polynomial has a repeated root, which we call \( r \). Here is another way to find the second linearly independent solution.

Because \( y = e^{rt} \) is a solution, we write \( (e^{rt})'' + p(e^{rt})' + q(e^{rt}) = 0 \).

We now differentiate both sides of this equation with respect to \( r \), assuming that the order of differentiation with respect to \( t \) and \( r \) may be interchanged. Show that the result is the equation \( (te^{rt})'' + p(te^{rt})' + q(te^{rt}) = 0 \), which implies that \( y = te^{rt} \) is also a solution.

**Quick Check Answers**

1. If \( y = e^{rt} \), then \( y' = re^{rt} = ry \) and \( y'' = r^2 e^{rt} = r^2 y \).

2. \( r^2 - 1 = 0 \); roots \( r = \pm 1 \) 3. \( r^2 + 2r + 1 = 0 \); solutions \( \{e^{-rt} \} \) 4. \( r^2 + 1 = 0 \); roots \( r = \pm i \)

5. \( p^2 - 1 = 0 \); roots \( p = \pm 1 \).

### 16.3 Linear Nonhomogeneous Equations

The homogeneous equations discussed in the previous section are used to model mechanical and electrical oscillators in the absence of external forces. Equally important are oscillators that are driven by external forces (such as an imposed load or a voltage source). When external forces are included in a differential equations model, the result is a nonhomogeneous equation, which is the subject of this section.

Solutions of a homogeneous oscillator equation are often called free oscillations. They represent the natural response of a system to an initial displacement or velocity—as specified by the initial conditions. For example, suppose a block on a spring is pulled away from its equilibrium position and then released. If no external forces act on the system, the motion of the block is determined by the initial conditions, the restoring force of the spring (which depends on the displacement of the block), and possibly friction (which often depends on the velocity of the block). In this case, we observe the system responding without outside influences.
If an external force is present, then the resulting response of the system is called a *forced oscillation*; it is a combination of the natural response and the effects of the force. In order to understand forced oscillations, we must know solutions of the nonhomogeneous equation (forces present) *and* solutions of the homogeneous equation (forces absent).

**Particular Solutions**

We now consider second-order equations of the form

\[ y''(t) + py'(t) + qy(t) = f(t), \]

where \( p \) and \( q \) are constants, and \( f \) is a specified function that is not identically zero on the interval of interest. Recall that any solution of this equation is called a *particular solution*, which we denote by \( y_p \). As proved in Section 16.1, there are infinitely many particular solutions of a given equation, all differing by a solution of the homogeneous equation. Also shown in Section 16.1 is that if the homogeneous equation

\[ y''(t) + py'(t) + qy(t) = 0 \]

has the linearly independent solutions \( y_1 \) and \( y_2 \), then the general solution of the nonhomogeneous equation is

\[ y = c_1y_1 + c_2y_2 + y_p, \]

where \( c_1 \) and \( c_2 \) are arbitrary constants. If initial conditions are given, they are used to determine \( c_1 \) and \( c_2 \). So the procedure for solving initial value problems associated with a nonhomogeneous equation consists of the following steps.

**PROCEDURE** Solving Initial Value Problems

Use the following steps to solve the initial value problem

\[ y'' + py' + qy = f(t), y(0) = A, y'(0) = B. \]

1. Find two linearly independent solutions \( y_1 \) and \( y_2 \) of the homogeneous equation \( y'' + py' + qy = 0 \).
2. Find any particular solution \( y_p \) of the nonhomogeneous equation \( y'' + py' + qy = f(t) \).
3. Form the general solution \( y = c_1y_1 + c_2y_2 + y_p \).
4. Use the initial conditions \( y(0) = A \) and \( y'(0) = B \) to determine the constants \( c_1 \) and \( c_2 \).

We already know how to find solutions of the homogeneous equation, so we now turn to methods for finding particular solutions.
Method of Undetermined Coefficients

In theory, the right-side function $f$ could be any continuous or piecewise continuous function. We limit our attention to three families of functions:

- polynomial functions: $f(t) = p_n(t) = a_n t^n + \cdots + a_1 t + a_0$.
- exponential functions: $f(t) = Ae^{at}$, and
- sine and cosine functions: $f(t) = A \cos bt + B \sin bt$.

For the right-side functions listed above, the best method for finding a particular solution is the method of undetermined coefficients. Several examples illustrate the method.

**EXAMPLE 1** Undetermined coefficients with polynomials Find the general solution of the equation $y'' - 4y = t^2 - 3t + 2$.

**SOLUTION** We follow the steps given above.

1. The associated homogeneous equation is $y'' - 4y = 0$. Using the methods of Section 16.2, the characteristic polynomial is $r^2 - 4$, which has roots $r = \pm 2$. Therefore, the general solution of the homogeneous equation is $y = c_1 e^{2t} + c_2 e^{-2t}$.

2. A particular solution is a function $y_p$, which, when added to multiples of its derivatives, equals $t^2 - 3t + 2$. Such a function must be a polynomial; in fact, it is a polynomial of degree two or less. Therefore, we use a trial solution of the form $y_p = At^2 + Bt + C$, where $A$, $B$, and $C$ are called undetermined coefficients that must be found. We now substitute the trial solution into the left side of the differential equation:

$$y_p'' - 4y_p = \left((At^2 + Bt + C)'' - 4(At^2 + Bt + C)\right) \text{ Substitute.}$$

$$= 2A - 4At^2 - 4Bt - 4C \quad \text{Differentiate.}$$

$$= -4At^2 - 4Bt + 2A - 4C. \quad \text{Collect terms.}$$

Equating the right and left sides of the differential equation gives the equation

$$-4At^2 - 4Bt + 2A - 4C = t^2 - 3t + 2.$$

In order for this equation to hold for all $t$, the coefficients of $t^2$, $t$, and $1$ on both sides of the equation must be equal. Equating coefficients, we find the following conditions for $A$, $B$, and $C$.

- $t^2$: $-4A = 1$
- $t$: $-4B = -3$
- $1$: $2A - 4C = 2$

The solution of this set of equations is found to be

$$A = -\frac{1}{4}, \quad B = \frac{3}{4}, \quad \text{and} \quad C = -\frac{5}{8}.$$

3. We now assemble the general solution of the nonhomogeneous equation; it is

$$y = c_1 e^{2t} + c_2 e^{-2t} - \frac{1}{4} t^2 + \frac{3}{4} t - \frac{5}{8}.$$

**Related Exercises 9–12**
EXAMPLE 2  Undetermined coefficients with exponentials  Find the general solution of the equation \( y'' + y' - 2y = 6e^{3t} - e^{-t} \).

SOLUTION  Again, we follow the steps given above.

1. The homogeneous equation \( y'' + y' - 2y = 0 \) has the characteristic polynomial \( r^2 + r - 2 \), whose roots are \( r = 1 \) and \( r = -2 \). Therefore, the general solution of the homogeneous equation is \( y = c_1 e^t + c_2 e^{-2t} \).

2. To find a particular solution, we reason much as we did in Example 1. A particular solution \( y_p \) and multiples of its derivatives must add up to \( 6e^{3t} - e^{-t} \). A trial solution in this case is \( y_p = Ae^{3t} + Be^{-t} \), where \( A \) and \( B \) must be found. We now substitute the trial solution into the differential equation:

\[
\begin{align*}
  y_p'' + y_p' - 2y_p &= (Ae^{3t} + Be^{-t})'' + (Ae^{3t} + Be^{-t})' - 2(Ae^{3t} + Be^{-t}) \\
  &= 9Ae^{3t} + Be^{-t} + 3(3Ae^{3t} - Be^{-t}) - 2(Ae^{3t} + Be^{-t}) \\
  &= e^{-t}(9A + 3A - 2A) + e^{-t}(B - B - 2B) \\
  &= 10Ae^{3t} - 2Be^{-t} \\
  &= 6e^{3t} - e^{-t}.
\end{align*}
\]

This calculation results in the equation \( 10Ae^{3t} - 2Be^{-t} = 6e^{3t} - e^{-t} \), which holds for all \( t \) only if coefficients of like terms on both sides of the equation are equal.

\[
\begin{align*}
  e^{3t}: & \quad 10A = 6 \\
  e^{-t}: & \quad -2B = -1
\end{align*}
\]

In this case, we see that \( A = \frac{3}{5} \) and \( B = \frac{1}{2} \).

3. The general solution of the nonhomogeneous equation now follows:

\[
y = c_1 e^t + c_2 e^{-2t} + \frac{3}{5}e^{3t} + \frac{1}{2}e^{-t}.
\]

Related Exercises 13–16

Quick Check 2  What trial solution should be used to find the particular solution of the equation \( y'' - 2y = 4e^{-2t} \) ?

EXAMPLE 3  Undetermined coefficients with sines and cosines  Solve the initial value problem \( y'' + 16y = 15 \sin t, y(0) = 1, y'(0) = 1 \).

SOLUTION  

1. The homogeneous equation \( y'' + 16y = 0 \) has the characteristic polynomial \( r^2 + 16 \), which has roots \( r = \pm 4i \). Therefore, the general solution of the homogeneous equation is \( y = c_1 \sin 4t + c_2 \cos 4t \).

2. Based on the form of the right-side function, a likely trial solution for the particular solution is \( y_p = A \sin t + B \cos t \), where the values of \( A \) and \( B \) must be determined. Substituting the trial solution into the differential equation gives

\[
\begin{align*}
  y_p'' + 16y_p &= (A \sin t + B \cos t)'' + 16(A \sin t + B \cos t) \\
  &= -A \sin t - B \cos t + 16(A \sin t + B \cos t) \\
  &= -A \sin t - B \cos t + 16A \sin t + 16B \cos t \\
  &= 15A \sin t + 15B \cos t \\
  &= 15 \sin t.
\end{align*}
\]

The condition \( 15A \sin t + 15B \cos t = 15 \sin t \), which must hold for all \( t \), is satisfied provided \( A = 1 \) and \( B = 0 \).

3. It follows that the general solution of the nonhomogeneous equation is

\[
y = c_1 \sin 4t + c_2 \cos 4t + \sin t.
\]

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4. In this example, we are given initial conditions, which are used to determine the constants $c_1$ and $c_2$. First notice that $y'(t) = 4c_1 \cos 4t - 4c_2 \sin 4t + \cos t$. The initial conditions now imply that

$$y(0) = c_2 = 1 \quad \text{and} \quad y'(0) = 4c_1 + 1 = 1, \text{or } c_1 = 0.\]$$

Therefore, the solution of the initial value problem is

$$y = \cos 4t + \sin t.$$

The solution of the initial value problem (red curve in Figure 16.13) is instructive. It contains one term from the solution of the homogeneous equation, representing the natural response of the system, and one term from the particular solution, representing the effect of the external force. The resulting response is a high-frequency wave $y = \cos 4t$ superimposed on a low-frequency wave $y = \sin t$ (dashed curve in Figure 16.13).

The calculation that led to the solution also has a useful lesson. The trial solution has a term $(B \cos t)$ that does not appear in the particular solution. The method of undetermined coefficients will always return a value of zero for the coefficient of an unnecessary term. So it is better to err on the side of including too many terms in the trial solution.

**Related Exercises 17–20**

**Quick Check 3** What trial solution should be used to find the particular solution of the equation $y'' - 2y = \cos 5t - 3 \sin 5t$?

**Example 4** Undetermined coefficients with combined functions Find the general solution of the equation

$$y'' + 6y' + 25y = 29e^{-t} \sin t.$$

**Solution**

1. The characteristic polynomial of the homogeneous equation is $r^2 + 6r + 25 = 0$, whose roots are $r = -3 \pm 4i$. Therefore, the general solution of the homogeneous equation is

$$y = c_1 e^{-3t} \sin 4t + c_2 e^{-3t} \cos 4t.$$

2. To find a particular solution, we surmise that a term of the form $e^{at} \sin bt$ or $e^{at} \cos bt$ on the right side of the equation requires a trial solution of the form $y_p = Ae^{at} \sin bt + Be^{at} \cos bt$. For this particular problem, we let $a = -1$ and $b = 1$. Differentiating and collecting terms, we find that

$$y_p' = -(A + B)e^{-t} \sin t + (A - B)e^{-t} \cos t \quad \text{and} \quad y_p'' = 2Be^{-t} \sin t - 2Ae^{-t} \cos t.$$

When the derivatives are substituted into the differential equation and simplifications are made, the result is the equation

$$(19A - 4B)e^{-t} \sin t + (4A + 19B)e^{-t} \cos t = 29e^{-t} \sin t.$$

We now equate coefficients of $e^{-t} \sin t$ and $e^{-t} \cos t$ on both sides of the equation to give the conditions on $A$ and $B$:

$$19A - 4B = 29 \quad \text{and} \quad 4A + 19B = 0.$$

A short calculation gives the values $A = \frac{19}{13}$ and $B = -\frac{4}{13}$.

Therefore, a particular solution is

$$y_p = \frac{19}{13} e^{-t} \sin t - \frac{4}{13} e^{-t} \cos t.$$
Combining the solution of the homogeneous equation and the particular solution, the general solution of the equation is

\[ y = c_1e^{-3t}\sin 4t + c_2e^{-3t}\cos 4t + \frac{19}{13}e^{-t}\sin t - \frac{4}{13}e^{-t}\cos t. \]

Taking \( c_1 = c_2 = 0 \), we see that the particular solution (Figure 16.14) is a rapidly damped wave.

![Figure 16.14](image)

**Quick Check 4** What trial solution should be used to find the particular solution of the equation \( y'' - 2y = 13e^{-2t}\cos 7t \)?

The four preceding examples demonstrate how the method of undetermined coefficients works. By now, you probably realize that if you choose an appropriate trial solution, then the method eventually produces a particular solution of an equation, although some persistence may be needed. Therefore, the crux of the method is choosing a good trial solution. Table 16.2 lists the trial solutions needed for commonly occurring right-side functions. In this table, lower case letters represent coefficients that are that upper, upper case letters are undetermined coefficients that must be found, and \( p_n \) is an \( n \)-th degree polynomial

\[ p_n(t) = a_n t^n + \cdots + a_1 t + a_0, \text{ where } a_n \neq 0. \]

**Table 16.2 Trial Solutions—Almost Always**

<table>
<thead>
<tr>
<th>Right-side function</th>
<th>Trial solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_n(t) )</td>
<td>( A_n t^n + \cdots + A_1 t + A_0 )</td>
</tr>
<tr>
<td>( e^{at} )</td>
<td>( Ae^{at} )</td>
</tr>
<tr>
<td>( \sin bt ) or ( \cos bt )</td>
<td>( A \sin bt + B \cos bt )</td>
</tr>
<tr>
<td>( p_n(t) e^{at} )</td>
<td>( (A_n t^n + \cdots + A_1 t + A_0)e^{at} )</td>
</tr>
<tr>
<td>( p_n(t) \sin bt ) or ( p_n(t) \cos bt )</td>
<td>( (A_n t^n + \cdots + A_1 t + A_0) \sin bt + (B_n t^n + \cdots + B_1 t + B_0) \cos bt )</td>
</tr>
<tr>
<td>( e^{at} \sin bt ) or ( e^{at} \cos bt )</td>
<td>( A e^{at} \sin bt + B e^{at} \cos bt )</td>
</tr>
</tbody>
</table>

If a right-side function is a sum of two or more functions (say, \( f = f_1 + f_2 \)), a particular solution must be found for each function in the sum. Then the particular solution for \( f \) is the sum of the particular solutions for \( f_1 \) and \( f_2 \) (Exercise 51).

**The Exceptions**

The trial solutions in Table 16.2 almost always work; however, there are a few exceptions. If the exceptions were insignificant, they might be overlooked. As we show in the next section, the exceptions correspond to an important physical phenomenon, so it’s necessary to investigate these special cases.
To find the general solution of the equation
\[ y'' + 9y = 4 \sin 3t, \]
we proceed as in previous examples. The characteristic polynomial of the homogeneous equation is \( r^2 + 9 \), which has roots \( r = \pm 3i \). Therefore, the general solution of the homogeneous equation is
\[ y = c_1 \sin 3t + c_2 \cos 3t. \]

To find the particular solution, we follow Table 16.2 and use the trial solution
\[ y_p = A \sin 3t + B \cos 3t. \]
Substituting this solution into the differential equation leads to
\[
y_p'' + 9y_p = \left( A \sin 3t + B \cos 3t \right)'' + 9 \left( A \sin 3t + B \cos 3t \right) = -9A \sin 3t - 9B \cos 3t + 9A \sin 3t + 9B \cos 3t - 9A \sin 3t - 9B \cos 3t = -9A \sin 3t + 9A \sin 3t + 9B \cos 3t - 9B \cos 3t = -9A + 9A \sin 3t + 9B - 9B \cos 3t \]
\[
= -9A + 9A \sin 3t + 9B - 9B \cos 3t = 0 \]
\[
\nequiv 4 \sin 3t, \quad \text{for all} \ t. \]

Ordinarily, at this stage of the calculation, we would match coefficients of \( \sin 3t \) and \( \cos 3t \) on both sides of the equation, and solve for \( A \) and \( B \). However, in this case, we find that \( y_p'' + 9y_p = 0 \), which means the trial solution failed. The failure occurred because a solution of the homogeneous equation appears on the right side of the equation.

Before explaining the remedy for this special case, let’s see what it means physically. Recall that solutions of the homogeneous equation represent the natural response of the system in the absence of external forces. When an external force is applied to a system that matches the natural response of the system, something unusual happens. As we discuss in the next section, this occurrence is called resonance.

When a solution of the homogeneous equation appears on the right side of the equation, the trial solution must be adjusted slightly. The adjusted trial solution is the trial solution given in Table 16.2 multiplied by \( t^s \), where \( s = 1 \) or \( s = 2 \) is chosen to ensure that no term in the trial solution is a solution of the homogeneous equation (Exercise 52).

Table 16.2 may be revised and presented concisely as follows.

**SUMMARY**  **Trial Solutions for**  \( y'' + py' + qy = f \)

To find the particular solution of the equation
\[ y'' + py' + qy = p_h(t)e^{rt}, \]
use the trial solution given in Table 16.2 multiplied by \( t^s \), where \( s \) is the smallest nonnegative integer that ensures that no term of \( y_p \) is a homogeneous solution.

**EXAMPLE 5**  **Special cases**  Find the general solution of the equation
\[ y'' + 9y = 4 \sin 3t. \]

**SOLUTION**  We notice that the right-side function \( 4 \sin 3t \) is a solution of the homogeneous equation. Therefore, we modify the standard trial solution \( y_p = A \sin 3t + B \cos 3t \) and instead use
\[ y_p = At \sin 3t + Bt \cos 3t. \]
Substituting into the differential equation, we see that
\[ y_p'' + 9y_p = (At \sin 3t + Bt \cos 3t)'' + 9(At \sin 3t + Bt \cos 3t) \]
\[ = A(6 \cos 3t - 9t \sin 3t) + B(-6 \sin 3t - 9t \cos 3t) \]
\[ y_p'' \]
\[ + 9At \sin 3t + 9Bt \cos 3t \]
\[ = -6B \sin 3t + 6A \cos 3t \]
\[ = 4 \sin 3t, \quad \text{Equals right side} \]
\[ 0 \]
\[ 0 \]
\[ \text{Simplify.} \]
\[ \text{Differentiate.} \]
\[ \text{Substitute.} \]

Notice the cancellation of terms leaves us with a linear combination of \( \sin 3t \) and \( \cos 3t \). We now use the equation \(-6B \sin 3t + 6A \cos 3t = 4 \sin 3t \) and match coefficients of \( \sin 3t \) and \( \cos 3t \). We conclude that \( A = 0 \) and \( B = \frac{2}{3} \). Therefore, the particular solution is \( y_p = -\frac{2}{3}t \cos 3t \), and the general solution of the equation is
\[ y = c_1 \sin 3t + c_2 \cos 3t - \frac{2}{3}t \cos 3t. \]

Figure 16.15 shows the particular solution \( (c_1 = c_2 = 0) \) and its significant property: Because of the multiplicative factor of \( t \) in the solution, the amplitude of this oscillation increases linearly with \( t \). The solution is wrapped in an envelope formed by two lines with slopes \( \pm \frac{2}{5} \).

**Quick Check 5** What trial solution should be used to find the particular solution of the equation \( y'' - 4y = 6e^{3t} \)?

In closing, we mention one remaining case that hasn’t been discussed. Consider the equation \( y'' + 4y' + 4y = 3e^{-2t} \). You can check that the characteristic polynomial for the homogeneous equation is \( r^2 + 4r + 4 = (r + 2)^2 \). Therefore, the characteristic polynomial has the repeated root \( r = -2 \) and the solution of the homogeneous equation (by Section 16.2) is \( y = c_1 e^{-2t} + c_2 te^{-2t} \). Now when we look for a particular solution, we see that a solution of the homogeneous equation appears on the right side of the equation. The trial solution cannot be \( y_p = Ae^{-2t} \) or \( y_p = Ate^{-2t} \), because both of these functions are solutions of the homogeneous equation. In this case, the correct trial solution is \( y_p = At^2 e^{-2t} \).
SECTION 16.3 EXERCISES

Review Questions
1. Explain how to find the general solution of the nonhomogeneous equation $y'' + py' + qy = f(t)$, where $p$ and $q$ are constants.
2. What trial solution would you use to find a particular solution of the equation $y'' + py' + qy = t^3 - t + 1$, where $p$ and $q$ are constants?
3. What trial solution would you use to find a particular solution of the equation $y'' - 8y = 2e^{-4t}$?
4. What trial solution would you use to find a particular solution of the equation $y'' - 2y = 2 \sin 3t$?
5. What trial solution would you use to find a particular solution of the equation $y'' - 2y = 6e^{-3t} \cos 4t$?
6. What trial solution would you use to find a particular solution of the equation $y'' - 2y = 1 + t + \cos 3t$?
7. What trial solution would you use to find a particular solution of the equation $y'' - 4y = t^2 e^{-2t}$?
8. What trial solution would you use to find a particular solution of the equation $y'' - 4y' + 4y = 6e^{2t}$?

Basic Skills
9–12. Undetermined coefficients with polynomials Find a particular solution of the following equations.
9. $y'' - 9y = 2t + 1$
10. $y'' - 2y' - 15y = 3t^3 - 2t - 4$
11. $y'' + 3y' - 18y = 2t^4 - t^2$
12. $y'' + 5y = 6t^3 - t^2$
13–16. Undetermined coefficients with exponentials Find a particular solution of the following equations.
13. $y'' - y = e^{-2t}$
14. $y'' + 3y = 5e^{-3t}$
15. $y'' - 4y' - 32y = 6e^{-3t}$
16. $y'' + 2y' - 8y = e^{-t} - 2e^t$
17–20. Undetermined coefficients with sines/cosines Find a particular solution of the following equations.
17. $y'' - y = 3 \sin 2t$
18. $y'' + 4y = 5 \cos 3t$
19. $y'' - y' - 6y = \sin t + 3 \cos t$
20. $y'' - 3y' - 4y = 2 \cos 2t - 3 \sin 2t$
21–28. Undetermined coefficients with combined functions Find a particular solution of the following equations.
21. $y'' - 4y = 2e^t - 1$
22. $y'' + y = \cos 2t + t^2$
23. $y'' + y' - 12y = 2te^{-t}$
24. $y'' + 4y = e^{-t} \cos t$
25. $y'' + 16y = t \cos 2t$
26. $y'' - 9y = (t^2 + 1) e^{-t}$
27. $y'' - y' - 2y = t(\cos t - \sin t)$
28. $y'' + 2y' - 4y = t^2 \cos t$

29–34. Undetermined coefficients with special cases Find a particular solution of the following equations.
29. $y'' - y = 3e^t$
30. $y'' + y = 3 \cos t$
31. $y'' + y' - 6y = 4e^{-3t}$
32. $y'' + 4y = \cos 2t$
33. $y'' + 5y' + 6y = 2e^{-2t}$
34. $y'' + 2y' + y = 4e^{-t}$

Further Explorations
35. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.
   a. To find a particular solution of the equation $y'' - 4y = t^3$, you should use the trial solution $y_p = At^3$.
   b. To find a particular solution of the equation $y'' + y' - 6y = \sin t$, you should use the trial solution $y_p = A \sin t$.
   c. To find a particular solution of the equation $y'' + 10y' + 25y = e^{5t}$, you should use the trial solution $y_p = Ae^{5t}$.
36–41. Initial value problems Find the general solution of the following equations and then solve the given initial value problem.
36. $y'' - 9y = 2e^{-t}, y(0) = 0, y'(0) = 0$
37. $y'' + y = 4 \sin 2t, y(0) = 1, y'(0) = 0$
38. $y'' + 3y' + 2y = 2e^{-t}, y(0) = 0, y'(0) = 1$
39. $y'' + 4y' + 5y = 12, y(0) = 1, y'(0) = -1$
40. $y'' - y = 2e^{-t} \sin t, y(0) = 4, y'(0) = 0$
41. $y'' + 9y = 6 \cos 3t, y(0) = 0, y'(0) = 0$
42–45. Longer calculations Find a particular solution of the following equations.
42. $y'' + 2y' + 2y = 5e^{-2t} \cos t$
43. $y'' + y' - y = 3t^2 - 3t^3 + t$
44. $y'' + 2y' + 5y = 8e^{-t} \cos 2t$
45. $y'' - y = 25e^{-t} \sin 3t$
46–49. Higher-order equations The method of undetermined coefficients extends naturally to higher-order constant coefficient equations. Find a particular solution of the following equations.
46. $y^{(n)} - y' = 8t^3$
47. $y'' - 8y = 7e^t$
48. \( y^{(4)} - y = 5 \sin 2t \)
49. \( y^{(4)} - 3y'' + 2y = 6te^{2t} \)

**Applications**

*Applications of Sections 16.2 and 16.3 are considered in Section 16.4.*

**50. Looking ahead: beats** One of the applications considered in Section 16.4 results in the initial value problem

\[ y'' + \omega_0^2 y = a \cos \omega t, \quad y(0) = 0, \quad y'(0) = 0, \]

where \( a, \omega, \omega_0 \) are positive real numbers.

a. Assume that \( \omega \neq \omega_0 \) and find the general solution of this equation.

b. Solve the initial value problem and show that the solution is

\[ y = \frac{a}{\omega_0^2 - \omega^2} \left( \cos \omega t - \cos \omega_0 t \right). \]

c. The case in which \( \omega \) and \( \omega_0 \) are nearly equal is of interest. Let \( \omega = 5 \) and \( \omega_0 = 6 \) and graph the solution on \([0, 6\pi]\). Describe the solution you see.

d. In order to identify the frequencies in the solution, use the identity

\[ \cos \omega_0 t - \cos \omega t = 2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2} \]

to rewrite the solution in terms of a product of two sine functions.

e. Now interpret the graph of part (c) and identify the two frequencies in the solution.

**Additional Exercises**

**51. Sum of particular solutions** Show that a particular solution of the equation \( y'' + py' + qy = f(t) + g(t) \) is \( y_p + y_{p2} \), where \( y_{p1} \) is a particular solution of \( y'' + py' + qy = f(t) \) and \( y_{p2} \) is a particular solution of \( y'' + py' + qy = g(t) \).

**52. Variation of parameters** Finding a particular solution when a homogeneous solution appears in the right-side function is handled using a method called *variation of parameters.* (This method is also used to find particular solutions of equations that cannot be handled by undetermined coefficients.) The following steps show how variation of parameters is used to find the particular solution of one specific equation.

a. Consider the equation \( y'' - y = e^t \). Show that the homogeneous solutions are \( y_1 = e^t \) and \( y_2 = e^{-t} \). Note that the right-side function is a homogeneous solution.

b. Assume a particular solution has the form

\[ y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = u_1(t)e^t + u_2(t)e^{-t}, \]

where the functions \( u_1 \) and \( u_2 \) are to be determined. Compute \( y_p' \) and impose the condition \( u_1'e^t + u_2'e^{-t} = 0 \) to show that \( y_p' = u_1'e^t - u_2'e^{-t} \).

c. Compute \( y_p'' \) and substitute it into the differential equation.

After canceling several terms, show that the equation reduces to \( u_1'e^t - u_2'e^{-t} = e^t \).

d. Parts (c) and (d) give two equations for \( u_1' \) and \( u_2' \). Eliminate \( u_2' \) and show that the equation for \( u_1 \) is \( u_1' = \frac{1}{2} \).

e. Solve the equation in part (d) for \( u_1 \).

f. Use part (c) to show that the equation for \( u_2 \) is \( u_2' = -\frac{1}{2}e^{2t} \).

g. Solve the equation in part (f) for \( u_2 \).

h. Now assemble the particular solution \( y_p(t) = u_1(t)e^t + u_2(t)e^{-t} \) and show that \( y_p = \frac{1}{2}e^t - \frac{1}{4}e^{-t} \).

**Quick Check Answers**

1. \( y_p = A + B \)  
2. \( y_p = Ae^{-5t} \)  
3. \( y_p = A \sin 5t + B \cos 5t \)  
4. \( y_p = Ae^{-2t} \sin 7t + Be^{-2t} \cos 7t \)  
5. \( y_p = Ae^{2t} \)

**16.4 Applications**

Throughout this chapter, we have alluded to many applications of second-order differential equations. We now have the tools needed to formulate and solve some of the second-order differential equations that arise in practical applications.

**Mechanical Oscillator Equations**

The term oscillator is used in several different ways. It may be a block bouncing on a spring, a pendulum on a grandfather clock, a wave on an oscilloscope, or a flowing and ebbing tide. In this section, we consider both mechanical and electrical oscillators, and because mechanical oscillators are easier to visualize, we begin with them. The important fact is that the differential equations that govern various oscillators are similar to one another in many ways.
The prototype mechanical oscillator is a block on a spring. Whether the spring is oriented vertically or horizontally, the relevant differential equation is the same. Consider the system shown in Figure 16.16 in which a block of mass \( m \) is attached to a massless spring that hangs from a rigid support. When the block hangs at rest, the spring is partially stretched and the system is at equilibrium; we denote this position of the block \( y = 0 \). The positive \( y \)-axis points downward in the direction of the gravitational force; that is, positive positions of the block are below the equilibrium position.

At time \( t = 0 \), the block may be pulled downward or pushed upward to a position \( y_0 \), and it may be given an initial velocity \( v_0 \). Therefore, we have initial conditions of the form \( y(0) = y_0 \) and \( y'(0) = v_0 \). Our aim is to formulate a differential equation that describes the position \( y(t) \) of the block after it is set in motion at time \( t = 0 \).

Newton’s second law of motion is used to describe the motion of ordinary objects. For one-dimensional motion, it says that

\[
m \cdot \text{acceleration} = \text{sum of forces}
\]

or more briefly \( my'' = F \), where \( F \) may depend on \( t, y \), and \( y' \).

In this section, we consider three forces that could act on the block: the restoring force of the spring, damping (or friction or resistance), and external forces. Let’s take them in order.

1. **Restoring force**: In many situations, the force that the spring exerts on the block is well modeled by Hooke’s law—an experimental result that applies when the amount of stretching or compression of the spring is small compared to the overall length of the spring. Suppose that the block attached to the spring has position \( y \). According to Hooke’s law, the spring exerts a force on the block that pulls or pushes the block toward the equilibrium position. This restoring force is given by \( F_s = -ky \), where \( k > 0 \) is a constant determined by the stiffness of the spring. Figure 16.17 shows how to interpret Hooke’s law. If \( y > 0 \), then the spring is stretched and \( F_s < 0 \), which means the force acts upward, pulling the block toward the equilibrium position. If \( y < 0 \), then the spring is compressed and \( F_s > 0 \), which means the force acts downward, pushing the block toward the equilibrium position. In either case, the spring works to “restore” the equilibrium.

2. **Damping force**: Resistance to the movement of the block can occur in many ways. The medium in which the block moves (perhaps air or oil) may act to resist motion. In some experiments, resistance may be provided by a dashpot (a piston and cylinder) attached to the block. In all these cases, the damping force acts in the direction opposite that of the motion. A common assumption is that the damping force is proportional to the velocity \( y' \). Therefore, we assume this force has the form \( F_d = -cy' \), where \( c > 0 \) measures the strength of the resistance.
3. External forces: Depending on the system being modeled, external forces $F_{\text{ext}}$ may arise in various ways. For example, in a vertically suspended spring-block system, the support of the spring may move, perhaps periodically with a specified amplitude and frequency. Whereas the restoring force depends on $y$ and the damping force depends on $y'$, the external force is constant or depends only on $t$.

With these three forces, Newton’s second law takes the form

$$m\ddot{y} = F_i + F_d + F_{\text{ext}} = -ky - cy' + F_{\text{ext}}.$$  

Each term in this equation has units of force, which we assume is measured in newtons (1 N = 1 kg m/s²). Therefore, the units of $k$ are N/m or kg/s², and the units of $c$ are kg/s. Dividing through the equation by $m$ and rearranging terms results in a familiar second-order constant-coefficient equation,

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = \frac{F_{\text{ext}}(t)}{m} \text{ or } y'' + by' + \omega_0^2y = f(t),$$

where $b = \frac{c}{m}$ and $\omega_0^2 = \frac{k}{m}$.

In the remainder of this section, we look at several versions of this equation. The following terminology helps organize these different cases.

- If the damping force is zero ($c = b = 0$), the motion is said to be undamped. Otherwise, the motion is damped.
- If no external forces are present ($F_{\text{ext}} = f = 0$), the system has free oscillations. Otherwise, the response consists of forced oscillations.

With this one equation of motion, there are four different categories of motion, each of which we investigate.

**Free Undamped Oscillations**

We now examine some of the behavior that is described by the general oscillator equation

$$y'' + by' + \omega_0^2y = f(t).$$

The easiest place to begin is with free ($f(t) = 0$) and undamped ($b = 0$) motion. This is the idealized case of a block that is given an initial displacement and/or velocity (as specified by the initial conditions) and then allowed to oscillate in the presence of the restoring force only. The equation of motion is simply

$$y'' + \omega_0^2y = 0, \text{ where } \omega_0^2 = \frac{k}{m}.$$  

Equations of this sort were solved in Section 16.2. The characteristic polynomial is $r^2 + \omega_0^2$, which has roots $r = \pm i\omega_0$. The general solution is

$$y = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t.$$  

We see that $\omega_0$ plays the role of an angular frequency; in this setting, it is the natural frequency of the oscillator. The solution describes a wave with constant amplitude and a period of $2\pi/\omega_0$; it is also an example of simple harmonic motion. Recall that a solution of the form $y = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t$ may always be expressed in amplitude-phase form $y = A \sin (\omega_0 t + \varphi)$.

**QUICK CHECK 2** Consider the function $y = 2 \sin 4.5t$. What is the period of the oscillation described by the function? How many oscillations does the graph have every 2π time units? How many oscillations does the graph have every time unit? <

**QUICK CHECK 1** Use the terms damped/undamped and free/forced to categorize the systems described by the following equations. (a) $y'' + 0.5y' + 4.3y = 0$, (b) $y'' + 9.2y = \sin 3t$.  

You can check that the units of $\omega_0$ are $s^{-1}$, which are the units of frequency. For example, $\omega_0 = 3$ means 3 oscillations per $2\pi$ time units, or $3/(2\pi)$ oscillations per unit time. If the angular frequency is $\omega_0$, then the period is $2\pi/\omega_0$ s.

See Section 16.2 for a review of the amplitude-phase form of a solution. In this case,

$$A = \sqrt{c_1^2 + c_2^2} \text{ and } \tan \varphi = \frac{c_2}{c_1}.$$
EXAMPLE 1  Free undamped oscillations  Suppose a block with mass 0.5 kg is attached to a spring, and the spring stretches 0.5 m to bring the system to equilibrium. The block is then pulled down an additional 0.25 m from its equilibrium position and released with an downward velocity of 0.1 m/s. Find the position of the block, for \( t \geq 0 \).

SOLUTION The first step is to determine the spring constant \( k \). Recall that the acceleration due to gravity is \( g = 9.8 \text{ m/s}^2 \). When the block is attached to the spring, it produces a force (equal to its weight) of \( mg = (0.5 \text{ kg}) \cdot (9.8 \text{ m/s}^2) = 4.9 \text{ N} \) on the spring. By Hooke’s law, this force equals \( ky \), where \( y = 0.5 \text{ m} \) is the distance the spring is stretched by the force. Therefore, the spring constant satisfies \( 0.5 = 4.9 \), which implies that \( k = 9.8 \text{ N/m} \). It follows that the natural frequency of the oscillator is \( \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{9.8}{0.5}} = 4.43 \text{ s}^{-1} \), which corresponds to a period of \( 2\pi/4.43 \approx 1.42 \text{ s} \).

The equation of motion has the general solution
\[
y = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t.
\]

We now use the initial conditions to evaluate \( c_1 \) and \( c_2 \). Noting that
\[
y' = c_1 \omega_0 \cos \omega_0 t - c_2 \omega_0 \sin \omega_0 t
\]
and using \( \omega_0 = 4.43 \), we find that
\[
y(0) = c_2 = 0.25 \quad \text{and} \quad y'(0) = c_1 \omega_0 = 0.1 \text{ or } c_1 = \frac{0.1}{\omega_0} = 0.023.
\]

Therefore, the solution of the initial value problem is
\[
y = 0.023 \sin 4.43 t + 0.25 \cos 4.43 t.
\]

The solution may also be expressed in amplitude-phase form as
\[
y = 0.251 \sin (4.43 t + 1.48).
\]

When the solution is written in this form, it is clear that the block oscillates with a constant amplitude of approximately \( 0.25 \) for all time (Figure 16.18a). Without damping, there is no loss of energy in the system, which is an idealized case.

![Figure 16.18](a)

The phase-plane graph of the solution (Figure 16.18b) shows clearly that the amplitude of the oscillation varies between approximately \( \pm 0.25 \), while the velocity varies between approximately \( \pm 1.1 \). The closed curve in the phase plane indicates that the solution is periodic with no change in the amplitude over time.

Related Exercises 7–12
Recall that we divided through Newton’s second law by the mass to produce an equation in terms of acceleration. Therefore, the external force must also be divided by \( m \).

Forced Undamped Oscillations: Beats

When an external force is included with undamped oscillations, some remarkable new effects appear. We limit our attention to external forces of the form \( F_{ext} = F_0 \cos \omega t \), where the amplitude \( F_0 \) and the frequency \( \omega \) of the force are specified. The resulting nonhomogeneous differential equation is

\[
y'' + \alpha^2_0 y = \frac{F_0}{m} \cos \omega t.
\]

Equations of this sort were considered in Section 16.3. As before, the solution of the homogeneous equation is

\[
y = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t.
\]

Using undetermined coefficients with the assumption that \( \omega \neq \omega_0 \), the particular solution (Exercise 38) is

\[
y_p = \frac{F_0}{m(\alpha_0^2 - \omega^2)} \cos \omega t.
\]

Therefore, the general solution for the forced undamped oscillator is

\[
y = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0}{m(\alpha_0^2 - \omega^2)} \cos \omega t.
\]

To isolate the effect of the external force, we consider the initial conditions \( y(0) = 0 \), \( y'(0) = 0 \), which implies that \( c_1 = 0 \) and \( c_2 = -\frac{F_0}{m(\alpha_0^2 - \omega^2)} \). The solution of the initial value problem becomes

\[
y = \frac{F_0}{m(\alpha_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t),
\]

which is the sum of two oscillatory functions with different frequencies.

The solution is easier to visualize when written using a trigonometric identity as (Exercise 39)

\[
y = \frac{2F_0}{m(\alpha_0^2 - \omega^2)} \left( \frac{\omega_0 - \omega}{2} \right) \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2}.
\]

This function is striking when \( \omega = \omega_0 \). In this case, \( \frac{2F_0}{m(\alpha_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \) is a low-frequency wave with amplitude \( \frac{2F_0}{m(\alpha_0^2 - \omega^2)} \) and a (large) period of \( \frac{2\pi}{(\omega_0 - \omega)/2} \), while \( \sin \frac{(\omega_0 + \omega)t}{2} \) is a high-frequency wave with amplitude 1 and a (small) period of \( \frac{2\pi}{(\omega_0 + \omega)/2} \). When these terms appear in a product, the result is a high-frequency carrier wave enclosed by low-frequency envelope waves (Figure 16.19), a phenomenon known as beats. The effect can be heard when two tuning forks with nearly equal frequencies are struck at the same time. When the phenomenon occurs in electrical circuits, it is called amplitude modulation.

Quick Check 3 | If the wave in Figure 16.19 were a sound wave (perhaps generated by two tuning forks), at what times would the sound be loudest? \( \diamond \)
EXAMPLE 2  **Forced undamped oscillations**  Suppose the spring-block system in Example 1 is attached to a device that moves the support up and down vertically, imparting to the system a force given by \( F_{\text{ext}} = \cos \omega t \). Determine the position \( y(t) \) of the block, for \( t \geq 0 \), with forcing frequencies \( \omega = 4 \) and \( \omega = 1 \), assuming \( y(0) = y'(0) = 0 \).

**SOLUTION**  From Example 1, we know that \( m = 0.5 \) kg, which implies \( \frac{2F_0}{m} = 4 \); the natural frequency is \( \omega_0 = 4.43 \text{ s}^{-1} \). Therefore, the solution of the initial value problem is approximately

\[
y = \frac{4}{19.6 - \omega^2} \sin \left( \frac{4.43 - \omega}{2} t \right) \sin \left( \frac{4.43 + \omega}{2} t \right).
\]

The solutions are shown in Figure 16.20 with \( \omega = 4 \) and \( \omega = 1 \). In the first case, the natural frequency is nearly equal to the forcing frequency and we see a pattern of beats. By contrast, when the two frequencies differ significantly, the resulting solution is a less organized superposition of two waves. Notice that beats produce a reinforcement of the two component waves and the amplitude of the oscillation is greater with beats.

![Figure 16.20](image)

**FIGURE 16.20**

**Forced Undamped Oscillations: Resonance**

The other interesting situation that arises with forced undamped oscillations occurs when \( \omega = \omega_0 \); in this case, the frequency of the external force exactly matches the natural frequency. The differential equation becomes

\[
y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t.
\]

We now have the special case discussed in Section 16.3 in which a homogeneous solution appears on the right side of the equation. Recall that to find a particular solution in this case, we use a trial solution of the form \( y_p = At \sin \omega_0 t + Bt \cos \omega_0 t \). Carrying out a familiar calculation with undetermined coefficients, we find the particular solution

\[
y_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.
\]

Therefore, the general solution of the equation is

\[
y = c_1 \sin \omega_0 t + c_2 \cos \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t.
\]
Suppose \( v + c \mathbf{r} = 0 \), \( A \mathbf{c} \mathbf{r} + b \mathbf{r} = 0 \), \( b = \frac{c}{m} > 0 \), and \( \omega_0 = \sqrt{\frac{k}{m}} \) is the natural frequency. The characteristic polynomial for this equation is \( r^2 + br + \omega_0^2 \), which has roots
\[
r = \frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2}.
\]

The critical feature of this solution is the term \( t \sin \omega_0 t \), which is an oscillatory function with an increasing amplitude (Figure 16.21). This term leads to a phenomenon, known as resonance, which cannot persist in actual physical systems. In a realistic system, either the integrity of the system is altered (for example, a shock absorber fails or a bridge collapses) or damping eventually limits the growth of the oscillations.

**Example 3** Forced undamped oscillations Suppose the spring-block system in Example 1 is attached to a device that moves the support, imparting to the system a force given by \( F_{ext} = \cos \omega t \), where \( \omega = \omega_0 = 4.43 \) is the natural frequency of the system. Determine the position \( y(t) \) of the block, for \( t \geq 0 \), assuming \( y(0) = y'(0) = 0 \).

**Solution** With \( F_0 = 1 \), \( m = 0.5 \), and \( \omega_0 = 4.43 \), the general solution becomes
\[
y = c_1 \sin 4.43t + c_2 \cos 4.43t + 0.23t \sin 4.43t.
\]

Imposing the initial conditions, a short calculation shows that \( c_1 = c_2 = 0 \). Therefore, the solution of the initial value problem is
\[
y = 0.23t \sin 4.43t.
\]

The solution (Figure 16.22a) displays the increasing amplitude associated with resonance. The outward spiral of the phase-plane graph of the solution (Figure 16.22b) also gives a vivid picture of the amplitude growth.

**Free Damped Oscillations**
Truly undamped oscillations do not occur in most physical systems because friction or resistance is always present in some form. We now turn to more realistic oscillators in which some type of damping is present. With no external forces, the governing equation is
\[
y'' + by' + \omega_0^2 y = 0,
\]
where \( b = \frac{c}{m} > 0 \) measures the strength of the damping and \( \omega_0 = \sqrt{\frac{k}{m}} \) is the natural frequency. The characteristic polynomial for this equation is \( r^2 + br + \omega_0^2 \), which has roots
\[
r = \frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2}.
\]
16.4 Applications

This problem is intriguing because three cases arise depending on the relative sizes of $b$ (damping) and $\omega_0$ (restoring force).

1. If $b^2 > 4\omega_0^2$, the damping force is dominant, and the roots $r_1$ and $r_2$ are real and distinct. Furthermore, notice that because $b > 0$ and $\sqrt{b^2 - 4\omega_0^2} < b$, both roots are negative. Therefore, we expect exponentially decaying solutions of the form

$$y = c_1e^{r_1t} + c_2e^{r_2t}, \text{ with } r_1 < 0 \text{ and } r_2 < 0.$$  

In this situation, often called **overdamping**, the damping is strong enough to suppress oscillations.

2. If $b^2 = 4\omega_0^2$, then the characteristic polynomial has a double root $r_1 = -\frac{b}{2}$. We now have solutions of the form

$$y = c_1e^{r_1t} + c_2te^{r_1t}, \text{ with } r_1 < 0.$$  

In this case, the system “attempts” to oscillate, but the oscillations quickly decay or don’t occur at all. This borderline case is called **critical damping**.

3. If $b^2 < 4\omega_0^2$, then the characteristic polynomial has complex roots of the form

$$r_{1,2} = -\frac{b}{2} \pm i\frac{\sqrt{4\omega_0^2 - b^2}}{2}.$$  

In this situation, the restoring force dominates the damping force. As a result, the solutions are oscillatory with decaying amplitudes. This case is called **underdamping**, and solutions have the form

$$y = e^{-\frac{bt}{2}}(c_1 \sin \omega t + c_2 \cos \omega t), \text{ where } \omega = \frac{\sqrt{4\omega_0^2 - b^2}}{2}.$$  

**EXAMPLE 4** Free damped oscillations Suppose the spring-block system in Example 1 ($\omega_0 = 4.43$) is modeled with damping effects included. Determine the position $y(t)$ of the block, for $t \geq 0$, with the following damping coefficients and initial conditions.

a. $c = 1$, $y(0) = 0$, $y'(0) = 1$

b. $c = 4.43$, $y(0) = 0$, $y'(0) = 3$

c. $c = 5$, $y(0) = 0$, $y'(0) = 5$

**SOLUTION**

a. With $c = 1$ and $m = 0.5$, we have $b = \frac{c}{m} = 2$. Recalling that the natural frequency of the oscillator is $\omega_0 = 4.43$, we find that $b^2 - 4\omega_0^2 \approx -74.5 < 0$, and we have the case of underdamping. The roots of the characteristic polynomial are the complex numbers

$$r_1 \approx -1 + i\frac{\sqrt{74.5}}{2} \approx -1 + 4.3i \quad \text{and} \quad r_2 \approx -1 - i\frac{\sqrt{74.5}}{2} \approx -1 - 4.3i,$$

which give the general solution

$$y = e^{-t}(c_1 \sin 4.3t + c_2 \cos 4.3t).$$

Note that

$$y' = -e^{-t}(c_1 \sin 4.3t + c_2 \cos 4.3t) + e^{-t}(4.3c_1 \cos 4.3t - 4.3c_2 \sin 4.3t).$$

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The zeros of a damped oscillatory function are equally spaced. However, we generally do not refer to the period of such a function. Some texts use quasiperiod to describe the distance between local maxima in this case.

Therefore, the initial conditions imply that
\[ y(0) = c_2 = 0 \text{ and } y'(0) = 4.3c_1 = 1 \text{ or } c_1 \approx 0.23. \]

The solution of the initial value problem is \( y = 0.23e^{-t} \sin 4.3t \) and its graph (Figure 16.23a) features damped oscillations. After the block has oscillated three times, the amplitude of the oscillation has been reduced by a factor of approximately 20. The inward spiral of the phase-plane graph also illustrates the decaying oscillations (Figure 16.23b).

![Figure 16.23](image)

b. With \( m = 0.5 \text{ and } c = \omega_0 = 4.43, \) we have \( b = 2c, \) which implies that \( b^2 - 4\omega_0^2 = 0. \) In this case, we have critical damping. The characteristic polynomial has the single repeated root \( r = \frac{-b}{2} = -4.43. \) Therefore, the general solution of the equation is \( y = c_1e^{-4.43t} + c_2te^{-4.43t}. \) Before using the initial conditions we compute
\[ y' = -4.43c_1e^{-4.43t} + c_2e^{-4.43t}(1 - 4.43t). \]
The initial conditions imply that
\[ y(0) = c_1 = 0 \text{ and } y'(0) = c_2 = 3. \]
The solution of the initial value problem becomes
\[ y = 3te^{-4.43t}. \]
The solution (Figure 16.24) illustrates the idea of critical damping. If the spring in this system is a shock absorber on a truck, a sudden jolt \( (y'(0) = 3) \) is quickly damped and the truck returns to its equilibrium position smoothly. Depending on the initial conditions, the object could overshoot the equilibrium position once, but the system does not undergo oscillations.

c. When the damping constant is increased to \( c = 5, \) we see another type of behavior. With \( m = 0.5 \text{ and } \omega_0 = 4.43, \) we have \( b = 2c = 10, \) which implies that \( b^2 - 4\omega_0^2 = 21.5 > 0. \) The roots of the characteristic polynomial are
\[ r = \frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2} \approx -2.68 \text{ and } -7.32, \]
and the general solution is
\[ y = c_1e^{-2.68t} + c_2e^{-7.32t}. \]
The initial conditions lead to the equations

\[ c_1 + c_2 = 0 \]

\[ -2.68c_1 - 7.32c_2 = 5, \]

which have the solutions \( c_1 = 1.08 \) and \( c_2 = -1.08 \). Therefore, the solution of the initial value problem is \( y = 1.08e^{-2.68t} - 1.08e^{-7.32t} \). The solution (Figure 16.25) decays quickly, at a rate determined by the least negative root. In this case of overdamping, the resistance dominates the restoring force, and there are no oscillations.

**Related Exercises 17–20**

**Forced Damped Oscillations**

The final case includes all the effects discussed so far: a restoring force, damping, and an external force. Some of the many features exhibited by such models are explored in the exercises. In this section, we focus on the behavior that is common to most problems in this category.

We now consider equations of the form

\[ y'' + by' + \omega_0^2y = F_0 \cos \omega t. \]

The first observation is that as long as \( b > 0 \) (damped oscillations), the characteristic polynomial always has either negative real roots or complex roots with negative real parts. In either case, the solution of the homogeneous equation consists of terms that decrease in magnitude as \( t \) increases. Furthermore, as long as \( \omega \neq \omega_0 \), the particular solution has the form \( A \sin \omega t + B \sin \omega t \). Therefore, the general solution looks like

\[ y = c_1y_1 + c_2y_2 + A \sin \omega t + B \sin \omega t. \]

Because the solution of the homogeneous equation decays as \( t \) increases, it is called a **transient solution**. It contains the effects of the initial conditions (that are used to determine \( c_1 \) and \( c_2 \)), but eventually it dies out. What remains as \( t \) increases is the **steady-state** particular solution, which is determined by the external force. In general, we see that as \( t \to \infty \), \( y(t) \to y_p(t) \).

Figure 16.26 shows the solution to the initial value problem:

\[ y'' + 2y' + 10y = 30 \cos t, \quad y(0) = 20, \quad y'(0) = -10. \]

The transient solution (dashed blue curve) persists until approximately \( t = 5 \), at which point the steady-state solution (dashed red curve) with the same frequency as the external force dominates. The solution to the initial value problem (solid curve) is the sum of the transient and steady-state solutions.

Forced damped oscillations occur frequently in electrical circuits, the topic that we now consider. **Related Exercises 21–24**

**Electrical Circuits**

Our survey of electrical circuits is brief, in part because all the mathematics has been done. As you will see, the differential equations that govern circuits are identical to mechanical oscillator equations. Only a change of terminology is required to complete this beautiful analogy.

A basic electrical circuits that displays a wide variety of behavior is the LCR circuit. As shown in Figure 16.27, the circuit has four components connected in series:

- a voltage source such as a battery or generator with output measured in volts,
- an inductor (coil) with an inductance of \( L \) henries,
- a resistor with a resistance of \( R \) ohms, and
- a capacitor with a capacitance of \( C \) farads.
We assume that initially there is no current flowing in the circuit and no charge on the capacitor. When the switch is closed at \( t = 0 \), a current \( I(t) \), measured in amperes, flows through the circuit and a charge \( Q(t) \), measured in coulombs, builds on the capacitor. The current is the rate of change of the charge; that is,

\[ I(t) = \frac{dQ}{dt}. \]

The differential equation that describes the current in the circuit comes directly from Kirchhoff’s voltage law: *In a closed circuit, the sum of the voltage drops across the inductor, resistor, and capacitor equals the applied voltage.* Elementary circuit theory also gives the voltage drops across the components.

- An inductor stores energy in a magnetic field and produces a voltage drop of \( L\frac{dI}{dt} \).
- A resistor dissipates energy and produces a voltage drop of \( RI \).
- A capacitor stores energy in an electric field and produces a voltage drop of \( Q/C \).

Letting \( E(t) \) denote the applied voltage, the equation for the circuit is

\[ L\frac{dI}{dt} + RI + \frac{1}{C}Q = E(t). \]

(1)

As it stands, this equation involves both the current \( I \) and the charge \( Q \). We can proceed in two ways. First, we can use the facts that \( Q'(t) = I(t) \) and \( Q''(t) = I'(t) \), substitute into equation (1), and write an equation for the charge:

\[ LQ'' + RQ' + \frac{1}{C}Q = E(t). \]

(2)

Often, the current is of more interest. So we can also differentiate both sides of equation (1) and use \( Q'(t) = I(t) \) to give an equation for the current:

\[ LI'' + RI' + \frac{1}{C}I = E'(t). \]

(3)

We consider both of these equations.

The assumption that initially there is no current flowing in the circuit and no charge on the capacitor means that the initial conditions for equation (2) are \( Q(0) = 0 \) and \( I(0) = Q'(0) = 0 \). The initial conditions for equation (3) must specify values of \( I(0) \) and \( I'(0) \). We already have \( I(0) = 0 \). To determine the initial value \( I'(0) \), we use equation (1) with \( t = 0 \) to find that

\[ LI'(0) + RI(0) + \frac{1}{C}Q(0) = E(0), \]

which implies that \( I'(0) = \frac{E(0)}{L} \).

Working first with the current equation (3), we divide the equation by \( L \) to obtain the initial value problem:

\[ I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t), \quad I(0) = 0, \quad I'(0) = \frac{E(0)}{L}. \]

Notice that this differential equation is second-order with constant coefficients—just like the mechanical oscillator equation. The remarkable analogy between the two equations is shown in Table 16.3.
Table 16.3

<table>
<thead>
<tr>
<th>Mechanical oscillator</th>
<th>Electrical circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$my'' + cy' + ky = f$</td>
<td>$Lq'' + Rq' + \frac{1}{C}q = E'$</td>
</tr>
</tbody>
</table>

Analogous terms

<table>
<thead>
<tr>
<th>Position $y$</th>
<th>Current $I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass $m$</td>
<td>Inductance $L$</td>
</tr>
<tr>
<td>Damping $c$</td>
<td>Resistance $R$</td>
</tr>
<tr>
<td>Spring constant $k$</td>
<td>Reciprocal capacitance $1/C$</td>
</tr>
<tr>
<td>External force $f$</td>
<td>Derivative of voltage source $E'$</td>
</tr>
</tbody>
</table>

Before solving a specific initial value problem, we can make some observations about the general solution. Letting

$$p = \frac{R}{L}, \quad q = \frac{1}{LC}, \quad \text{and} \quad f(t) = \frac{1}{L}E'(t),$$

the equation appears as

$$I'' + pI' + qI = f(t).$$

The characteristic polynomial is $r^2 + pr + q$, and because $p = \frac{R}{L} > 0$, the characteristic polynomial always has either negative real roots or complex roots with negative real parts (just as in the case of forced damped oscillators). Therefore, the solution of the homogeneous equation is a transient solution that decays in time. The steady-state solution is the particular solution, which depends on the form of the right-side function $f$

**EXAMPLE 5**  LCR circuit with constant voltage  A circuit consists of a 1-henry inductor, a 50-ohm resistor, and a 1/7025-farad capacitor, connected in series, all driven by a constant 800-volt source. Determine the current in the circuit at all times after the switch is closed.

**SOLUTION** Let $L = 1$, $R = 50$, $1/C = 7025$, and $E(t) = 800$. Observing that $E'(t) = 0$, the equation becomes

$$I'' + 50I' + 7025I = 0,$$

which is a homogeneous equation. The characteristic polynomial $r^2 + 50r + 7025$ has roots

$$r = \frac{-50 \pm \sqrt{2500 - 4 \cdot 7025}}{2} = -25 \pm 80i.$$ 

Therefore, the general solution of the equation is

$$I = e^{-25t}(c_1 \sin 80t + c_2 \cos 80t).$$

Notice that this circuit is analogous to an underdamped oscillator. The initial conditions are

$$I(0) = 0 \quad \text{and} \quad I'(0) = \frac{E(0)}{L} = 800.$$ 

We first compute

$$I'(t) = -25e^{-25t}(c_1 \sin 80t + c_2 \cos 80t) + e^{-25t}(80c_1 \cos 80t - 80c_2 \sin 80t),$$
which implies that \( I'(0) = -25c_2 + 80c_1 \). The initial conditions give the values of \( c_1 \) and \( c_2 \):
\[
I(0) = c_2 = 0 \quad \text{and} \quad I'(0) = -25c_2 + 80c_1 = 800, \quad \text{or} \quad c_1 = 10.
\]

We conclude that the solution of the initial value problem is
\[
I = 10 e^{-25t} \sin 80t.
\]

The graph of the current shows a high frequency wave that is quickly damped (Figure 16.28a). The charge on the capacitor (obtained by integrating the current) shows a few oscillations of a transient response before reaching a steady-state level of about 0.11 C within 0.2 seconds (Figure 16.28b).

**FIGURE 16.28**

**EXAMPLE 6  LCR circuit with oscillatory voltage**  Consider an LCR circuit with a 1-henry inductor, a 50-ohm resistor, and a 1/2500-farad capacitor, driven by a voltage of \( E(t) = 1021 \cos 60t \). Determine the charge on the capacitor at all times after the switch is closed.

**SOLUTION** We now work with the charge equation (2). With \( L = 1 \), \( R = 50 \), and \( 1/C = 2500 \), the relevant initial value problem is
\[
Q'' + 50Q' + 2500Q = 1021 \cos 60t, \quad Q(0) = Q'(0) = 0.
\]

The characteristic polynomial is \( r^2 + 50r + 2500 \), which has roots
\[
r = -25 \pm \sqrt{2500 - 4 \cdot 2500} = -25 \pm 25 \sqrt{3}i.
\]

The solution of the homogeneous equation consists of oscillatory functions with decreasing amplitudes:
\[
Q_h = e^{-25t}(c_1 \sin 25 \sqrt{3}t + c_2 \cos 25 \sqrt{3}t).
\]

To find the particular solution using undetermined coefficients, the appropriate trial solution (determined by the right-side function) is
\[
Q_p = A \sin 60t + B \cos 60t.
\]

Substituting the trial solution and carrying out the necessary calculations, we find that \( A = \frac{3}{10} \) and \( B = -\frac{11}{100} \). Therefore, the general solution of the equation is
\[
Q = e^{-25t}(c_1 \sin 25 \sqrt{3}t + c_2 \cos 25 \sqrt{3}t) + \frac{3}{10} \sin 60t - \frac{11}{100} \cos 60t.
\]
Finally, we impose the initial conditions \( Q(0) = Q'(0) = 0 \). After more calculations (greatly assisted by a computer algebra system), the solution of the initial value problem emerges:

\[
Q = e^{-25t} \left( -\frac{61}{100\sqrt{3}} \sin 25\sqrt{3}t + \frac{11}{100} \cos 25\sqrt{3}t \right) + \frac{3}{10} \sin 60t - \frac{11}{100} \cos 60t.
\]

Far more informative is the graph of the charge, shown in Figure 16.29a. The transient part of the solution has an angular frequency of \( \omega = 25\sqrt{3} \), but it is quickly damped. The steady-state solution with constant amplitude and an angular frequency of \( \omega = 60 \) (period \( 2\pi/60 \approx 0.1 \)) is all that remains for \( t > 0.1 \). Similarly, the current \( I(t) = Q'(t) \) shows a brief transient solution followed by the steady-state solution with constant amplitude and frequency \( \omega = 60 \) (Figure 16.29b). We see that the presence of a variable voltage maintains an oscillatory current in the circuit and charge on the capacitor.

![Graph of charge and current](image)

**SECTION 16.4 EXERCISES**

**Review Questions**

1. Explain the meaning of the words damped, undamped, free, and forced as they apply to an oscillator system.
2. In the models discussed in this section, under what conditions do damped oscillations occur?
3. In the models discussed in this section, under what conditions do beats occur?
4. In the models discussed in this section, under what conditions does resonance occur?
5. Explain the steps required to solve an initial value problem for forced damped oscillations.
6. Describe the analogies between a model for a forced damped oscillator and a model for an LCR circuit.

**Basic Skills**

7. **Free undamped oscillations** Solve the initial value problem associated with the following experiments.

   A 1-kg block hangs on a spring with spring constant \( k = 1.44 \text{ N/m} \). The block is pulled down 0.75 m and released with no initial velocity. Assuming no resistance or external forcing, find the position of the block at all times after it is released. Graph and describe the motion of the block, and give its period of oscillation.

8. A 1.5-kg block hangs on a spring with spring constant \( k = 3.375 \text{ N/m} \). The block is pulled down 0.25 m and released with an upward velocity of 0.2 m/s. Assuming no resistance or external forcing, find the position of the block at all times after it is released. Graph and describe the motion of the block, and give its period of oscillation.

9. When a 2-kg block is attached to a spring, it stretches the spring 2.45 m. The block is lifted 0.3 m above its equilibrium position and released with no initial velocity. Assuming no resistance or external forcing, find the position of the block at all times after it is released. Graph and describe the motion of the block, and give its period of oscillation.

10. A 0.5-kg block hangs on a spring and stretches the spring 0.49 m. The block is pulled down 0.6 m and released with a downward velocity of 0.25 m/s. Assuming no resistance or external forcing, find the position of the block at all times after it is released. Graph and describe the motion of the block, and give its period of oscillation.
11. Consider a pendulum consisting of a bob attached by a massless rigid rod to a frictionless pivot that swings in a plane. Assuming no resistance or external forcing, the angular displacement of the pendulum may be approximated by the equation \( \ddot{\theta} + \omega_0^2 \theta = 0 \), where \( \omega_0 = \sqrt{\frac{g}{\ell}} \) is the length of the rod, and \( g = 9.8 \text{ m/s}^2 \) is the acceleration due to gravity (see Exercise 47). Assume a pendulum of length \( \ell = 0.98 \text{ m} \) is lifted 0.25 radian from its vertical equilibrium position and released with no initial velocity. Find the angular displacement of the pendulum at all times after it is released. Graph and describe the motion of the bob, and give its period of oscillation.

12. Referring to Exercises 11 and 47, consider a pendulum of length \( \ell = 0.49 \text{ m} \) that is lifted 0.25 radian from its vertical equilibrium position and released with an initial velocity of \( \dot{\theta}(0) = 0.1 \text{ rad/s} \). Find the angular displacement of the pendulum at all times after it is released. Graph and describe the motion of the bob, and give its period of oscillation.

13–16. Forced undamped oscillations Solve the initial value problem associated with the following experiments.

13. A 2-kg block hangs on a spring with spring constant \( k = 5.12 \text{ N/m} \). The support of the spring vibrates and produces an external force of \( F_{\text{ext}} = 4 \cos \omega t \). Assume no initial displacement and velocity of the block, and no damping. Graph the solution in two cases: \( \omega = 1.5 \) and \( \omega = 4 \). In which case does the solution have beats?

14. A 0.5-kg block hangs on a spring with spring constant \( k = 4.5 \text{ N/m} \). The support of the spring vibrates and produces an external force of \( F_{\text{ext}} = 2 \cos \omega t \). Assume no initial displacement and velocity of the block, and no damping. Graph the solution in two cases: \( \omega = 3.1 \) and \( \omega = 0.5 \). In which case does the solution have beats?

15. A 0.25-kg block hangs on a spring with spring constant \( k = 4.0 \text{ N/m} \). The support of the spring vibrates and produces an external force of \( F_{\text{ext}} = \sin \omega t \). Assume no initial displacement and velocity of the block, and no damping. Graph the solution in two cases: \( \omega = 4 \) and \( \omega = 1 \). In which case does the solution have resonance?

16. A 4/3-kg block hangs on a spring with spring constant \( k = 12.0 \text{ N/m} \). The support of the spring vibrates and produces an external force of \( F_{\text{ext}} = 9 \cos \omega t \). Assume no initial displacement and velocity of the block, and no damping. Graph the solution in two cases: \( \omega = 3 \) and \( \omega = 0.5 \). In which case does the solution have resonance?

17–18. Free damped oscillations Solve the initial value problem associated with the following experiments.

17. A 0.3-kg block hangs on a spring with spring constant \( k = 30 \text{ N/m} \). Air resistance slows the motion of the block with a damping coefficient of \( c \text{ kg/s} \). Solve an initial value problem with the following values of \( c \) and initial conditions \( y(0) = 1, \dot{y}(0) = 0 \). Graph each solution and state whether the problem exhibits underdamping, overdamping, or critical damping.

\[
\begin{align*}
a. & \quad c = 4.8 \text{ kg/s} \\
b. & \quad c = 6 \text{ kg/s} \\
c. & \quad c = 7.5 \text{ kg/s}
\end{align*}
\]

18. Three 10-kg blocks hang on springs with spring constant \( k \). Friction in each system is modeled with a damping coefficient of \( c = 40 \text{ kg/s} \). Solve an initial value problem with the following values of \( k \) and initial conditions \( y(0) = 2, \dot{y}(0) = 0 \). Graph each solution and state whether the problem exhibits underdamping, overdamping, or critical damping.

\[
\begin{align*}
a. & \quad k = 30 \text{ N/m} \\
b. & \quad k = 40 \text{ N/m} \\
c. & \quad k = 62.5 \text{ N/m}
\end{align*}
\]

19. Designing a shock absorber A shock absorber must bear a load of 250 kg (one-fourth the mass of the car). It is estimated that friction in the system can be modeled with a damping constant of \( c = 500 \text{ kg/s} \).

\[
\begin{align*}
a. & \quad \text{Find the value of the spring constant that provides critical damping for the system.} \\
b. & \quad \text{Solve the resulting differential equation subject to the initial conditions}\ y(0) = 0, y'(0) = 0.1 \text{ that model a sudden jolt.} \\
c. & \quad \text{Graph the solution in part (b). What is the maximum displacement of the car according to this model?} \\
d. & \quad \text{Determine the effect of increasing the value of } k \text{ in part (a) by 50% and by 100%.} \\
e. & \quad \text{What is the effect of decreasing the value of } k \text{ in part (a) by 50%?}
\end{align*}
\]

20. Designing a suspension system A spring in a suspension system supports a load of 400 kg and has a spring constant \( k = 324 \text{ N/m} \). The damping constant \( c \) in the system can be varied by embedding the spring in a piston filled with fluid.

\[
\begin{align*}
a. & \quad \text{Find the value of the damping constant } c \text{ that gives the system critical damping.} \\
b. & \quad \text{Solve the resulting differential equation subject to the initial conditions}\ y(0) = 0, y'(0) = 0.1 \text{ that model a sudden jolt.} \\
c. & \quad \text{Graph the solution in part (b). What is the maximum displacement of the load according to this model?} \\
d. & \quad \text{Determine the effect of increasing the damping constant to} c = 720\sqrt{2} \text{ and decreasing the damping constant to} c = 360\sqrt{3} \text{ in this model.}
\end{align*}
\]

21–24. Forced damped oscillations

21. A 1-kg block hangs from a spring with spring constant \( k = 5 \text{ N/m} \). A dashpot connected to the block introduces friction with damping constant \( c = 2 \text{ kg/s} \). The support of the spring vibrates and produces an external force of \( F_{\text{ext}} = 4 \cos 2t \). Assume the block is initially at rest at the equilibrium position and released with no initial velocity.

\[
\begin{align*}
a. & \quad \text{Find and graph the position of the block after it is released.} \\
b. & \quad \text{Identify the transient and steady-state solutions, and graph them with the solution in part (a).} \\
c. & \quad \text{After approximately how many seconds does the transient solution become negligible?}
\end{align*}
\]

22. A 20-kg block hangs from a spring with spring constant \( k = 180 \text{ N/m} \). The block oscillates in a fluid that produces resistance with a damping constant \( c = 80 \text{ kg/s} \). The support of the spring vibrates and produces an external force of \( F_{\text{ext}} = 3 \sin t \). Assume the initial position of the block is one meter above the equilibrium position and it is released with zero initial velocity.

\[
\begin{align*}
a. & \quad \text{Find and graph the position of the block after it is released.} \\
b. & \quad \text{Identify the transient and steady-state solutions, and graph them separately.} \\
c. & \quad \text{After approximately how many seconds does the transient solution become negligible?}
\end{align*}
\]
23. Consider an oscillator described by the equation
\[ y'' + y' + \frac{5}{4}y = 2 \cos at. \]

a. Find the general solution of the equation.
b. Find the solution of the initial value problem with 
\[ y(0) = y'(0) = 0. \]
c. Identify the transient and steady-state components of the solution.
d. Graph the solution of the initial value problem with 
\[ \omega = \frac{1}{7}, \] and 
\[ \frac{3}{2}. \] Which forcing frequency produces the steady-state oscillation with the greatest amplitude?

24. Consider an oscillator described by the equation
\[ y'' + 2y' + (\omega_0^2 + 1)y = 2 \cos t. \]

a. Find the general solution of the equation.
b. Find the solution of the initial value problem with 
\[ y(0) = y'(0) = 0. \]
c. Identify the transient and steady-state components of the solution.
d. Graph the solution of the initial value problem with 
\[ \omega_0 = 1, 2, \] and 
\[ 3. \] Which natural frequency produces the steady-state oscillation with the greatest magnitude?

25. An RC circuit Show that the charge on the capacitor of a circuit without an inductor satisfies the equation 
\[ R C' + \frac{1}{C} Q = E(t) \] with \( L = 0 \), it is an RC circuit. Find the charge on the capacitor in a circuit with a 50-ohm resistor and a 0.001-farad capacitor when it is driven by a constant 50-volt source. Does the charge reach a steady-state value? If so, what is it?

26. An RL circuit Show that the current in a circuit without a capacitor satisfies the equation 
\[ L I' + RI = E(t). \] Find the current in a circuit with a 200-ohm resistor and a 2-henry inductor when it is driven by a constant 100-volt source. Does the current reach a steady-state value? If so, what is it?

27–32. LCR circuits

27. An LCR circuit has a 10-ohm resistor, a 0.1-henry inductor, and a 1/240-farad capacitor connected in series to a constant 100-volt battery.

a. Find the current in the circuit assuming that \( I(0) = Q(0) = 0. \)
b. Identify the transient and steady-state currents.

28. The circuit in Exercise 27 (10-ohm resistor, a 0.1-henry inductor, and a 1/240-farad capacitor) is connected in series to a voltage source with output \( E(t) = 100 \sin 150t. \)

a. Find the current in the circuit assuming that \( I(0) = Q(0) = 0. \)
b. Identify the transient and steady-state currents.
c. Graph the steady-state, transient, and total current, for \( t \approx 0. \)

29. An LCR circuit has an 80-ohm resistor, a 0.5-henry inductor, and a 1/3230-farad capacitor connected in series to a constant 200-volt battery.

a. Find the current in the circuit assuming that \( I(0) = Q(0) = 0. \)
b. Identify the transient and steady-state currents.

30. An LCR circuit has an 80-ohm resistor, a 0.5-henry inductor, and a 1/3232-farad capacitor connected in series to a voltage source with output \( E(t) = 90.5 \sin 64t. \)

a. Find the current in the circuit assuming that \( I(0) = Q(0) = 0. \)
b. Identify the transient and steady-state currents.
c. Graph the steady-state, transient, and total current, for \( t \approx 0. \)

31. Find the charge on the capacitor and the current in an LCR circuit in which an 18-ohm resistor, a 2-henry inductor, and a 1/50-farad capacitor are connected in series to a voltage source with a constant 120-volt output. Assume \( Q(0) = Q'(0) = 0. \)

32. Find the charge on the capacitor and the current in an LCR circuit in which a 4-ohm resistor, a 0.5-henry inductor, and a 2/25-farad capacitor are connected in series to a voltage source with output \( E(t) = 30 \sin t. \) Assume \( Q(0) = Q'(0) = 0. \)

**Further Explorations**

33. Explain why or why not Determine whether the following statements are true and give an example or a counterexample.

a. An oscillator with damping cannot have a constant-amplitude position function.
b. Resonance occurs only with forced oscillations.
c. Resonance as it is defined in the text (solutions of the form \( y = At \sin \omega t \) or \( y = At \cos \omega t \)) cannot occur if there is damping in the system.
d. An LC circuit \( (R = 0) \) has damped oscillations.
e. Beats can occur in a system with damping.
f. The model discussed in the text for damped oscillators with periodic forcing always has a solution consisting of a decaying transient component and a non-decaying steady-state component.

34–35. Transient vs. steady-state Consider the following initial value problems.

a. Solve the initial value problem, and identify the transient and steady-state solutions.
b. Graph the solution of the initial value problem, the transient solution, and the steady-state solution.
c. Approximately how long does it take for the transient solution to disappear? Confirm that as \( t \) increases, the solution of the initial value problem approaches the steady-state solution.

34. \[ y'' + 2y' + 5y = 10 \cos t, \ y(0) = 1, \ y'(0) = 0 \]
35. \[ y'' + \frac{1}{2}y' + \frac{17}{16}y = 65 \sin t, \ y(0) = 0, \ y'(0) = 0 \]

36–37. All transient solutions die Consider the following oscillator equations.

a. Find the general solution of the equation and identify the transient and steady-state solutions.
b. Set \( c_3 = 0 \) and \( c_1 = -2, -1, 0, 1, \) and \( 2 \) in the general solution.
For each value of \( c_1 \), graph the solution and external force function on the same set of axes.
c. Describe the relationship between the solutions (for each value of \( c_1 \)) and the external force function.

36. \[ y'' + 2y' + 2y = 130 \sin 4t \]
37. $y'' + 3y' + 2y = 170 \cos 4t$

38. Forced undamped solution Show that a particular solution of the equation $y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t$ is $y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$.

39. Beats solution Recall the identity $\cos A - \cos B = 2 \sin \left( \frac{B - A}{2} \right) \sin \left( \frac{B + A}{2} \right)$.

**a.** Show that $\cos \omega t - \cos \omega_0 t = 2 \sin \left( \frac{\omega_0 - \omega}{2} \right) t \sin \left( \frac{\omega_0 + \omega}{2} \right) t$.

**b.** Graph the functions on both sides of the equation in part (a) with (i) $\omega_0 = 10$, $\omega = 9$ and (ii) $\omega_0 = 10$, $\omega = 2$ to verify the identity. In which case do you see beats?

40. Analysis of the forced damped oscillation equation Consider the equation $m y'' + c y' + ky = F_0 \cos \omega t$, which describes the motion of a forced damped oscillator. Assume all the parameters in the equation are positive.

**a.** Explain why the solutions of the homogeneous equation decay in time.

**b.** Show that a particular solution is $y_p = A \sin \omega t + B \cos \omega t$, where $A = \frac{c o F_0}{(c o)^2 + k - m \omega^2}$ and $B = \frac{c k F_0}{(c o)^2 + k - m \omega^2}$.

**c.** Using the amplitude-phase form of a solution, show that $y_p = A \sin \omega t + B \cos \omega t = C \sin (\omega t + \varphi)$, where $C^* = \sqrt{A^2 + B^2}$ and $\tan \varphi = \frac{B}{A}$.

**d.** Show that $C^* = \frac{F_0}{\sqrt{c o^2 + m^2(c o^2 - \omega_0^2)}}$, where $\omega_0 = \frac{k}{m}$.

**e.** What is the relationship between the forcing frequency $\omega$ and the natural frequency $\omega_0$ that produces the largest amplitude $C^*$? Explain why this result is analogous to resonance in the case of forced undamped motion.

**f.** Let $m = c = F_0 = 1$ and $\omega_0 = 3$. Graph the amplitude $C^*$ as a function of $\omega$. Describe how $C^*$ varies with respect to $\omega$.

41. Impedance Use the result of Exercise 40d and write the amplitude in terms of circuit parameters; that is, replace $m$ by $L$, $c$ by $R$, $o_0^2$ by $\frac{1}{CL}$, and $F_0$ by $oE_0$. Express the amplitude of the steady-state current as $C^* = \frac{E_0}{\sqrt{R^2 + \left( oL - \frac{1}{oC} \right)^2}}$.

The denominator in this expression is called the **impedance** (with units of ohms). Show that the impedance is minimized (which maximizes $C^*$) when $o^2 = \frac{1}{CL}$, which is the analog of $\omega_0 = \omega$.

Maximizing $C^*$ (minimizing the impedance) often gives the best reception in a radio receiver.

42. Gravity in the vertical oscillator In this derivation, we show why the gravitational force does not appear in the equation for a vertically suspended oscillator.

**a.** Consider the figure in which a block hangs at rest from a spring. The spring is stretched an amount $\Delta y$ by the weight $mg$ of the block. We denote this equilibrium position of the block $y = 0$. When the system is at equilibrium, explain why $mg = k \Delta y$, where $k$ is the spring constant.

**b.** Suppose the block is pulled down a distance $y$ from its equilibrium position. We now write Newton’s second law using the end of the unstretched spring as a reference point, assuming no damping or external force and assuming the $y$-axis points downward. Explain why $my'' = -k(y + \Delta y) + mg$.

**c.** Use the equilibrium relationship $mg = k \Delta y$, to express the equation of motion as $my'' = -ky$, which is the equation derived earlier. Notice that the gravitational force does not appear.

## Applications

43–46. Horizontal oscillators The equation of motion for a spring-block system that lies on a horizontal surface (see figure) is the same as the equation of motion for a vertically suspended system. As before, $m$ is the mass of the block, $k$ is the spring constant, $c$ is the damping coefficient (perhaps due to friction as the block slides on the surface), and $F_{ext}$ is an external force. We let $x(t)$ be the position of the block at time $t$, where $x$ increases to the right and $x = 0$ is the position of the block at which the spring is neither stretched nor compressed.

**a.** Find the position of the block in the following situations.

**b.** Graph the position function.

**c.** Describe the type of motion you observe.
43. \( m = 2 \text{ kg}, \ c = 6 \text{ kg/s}, \ k = 8 \text{ N/m}, \)
\[ F_{\text{ext}} = 0, \ x(0) = 2, \ x'(0) = -1 \]

44. \( m = 4 \text{ kg}, \ c = 8 \text{ kg/s}, \ k = 140 \text{ N/m}, \)
\[ F_{\text{ext}} = 0, \ x(0) = -2, \ x'(0) = 1 \]

45. \( m = 4 \text{ kg}, \ c = 4 \text{ kg/s}, \ k = 17 \text{ N/m}, \)
\[ F_{\text{ext}} = 148 \sin t, \ x(0) = 0, \ x'(0) = 0 \]

46. \( m = 8 \text{ kg}, \ c = 16 \text{ kg/s}, \ k = 136 \text{ N/m}, \)
\[ F_{\text{ext}} = 130 \cos t, \ x(0) = 0, \ x'(0) = 0 \]

47. **The pendulum equation** A pendulum consisting of a bob of mass \( m \) swinging on a massless rod of length \( \ell \) can be modeled as an oscillator (see figure). Let \( \theta(t) \) be the angular displacement of the pendulum \( t \) seconds after it is released (measured in radians). Assuming that the only force acting on the bob is the gravitational force, we write Newton’s second law in the direction of motion (perpendicular to the rod). Notice that the distance along the arc of the swing is \( s(t) = \ell \theta(t) \), so the velocity of the bob is \( s'(t) = \ell \theta'(t) \) and the acceleration is \( s''(t) = \ell \theta''(t) \).

![Circular arc](image)

\[ mg \]

**a.** Considering only the component of the force in the direction of motion, explain why Newton’s second law is
\[ m\theta''(t) = -mg \sin \theta(t) \]
where \( g = 9.8 \text{ m/s}^2 \) is the acceleration due to gravity.

**b.** Write this equation as \( \theta'' + \omega_0^2 \sin \theta = 0 \), where \( \omega_0 = \frac{g}{\ell} \).

**c.** Notice that this equation is nonlinear. It can be linearized by assuming that the angular displacements are small (\( \theta \ll 1 \)) and using the approximation \( \sin \theta = \theta \). Show that the resulting linear pendulum equation is \( \theta'' + \omega_0^2 \theta = 0 \).

**d.** Express the period of the pendulum in terms of \( g \) and \( \ell \). If the length of the pendulum is increased by a factor of 2, by what factor does the period change?

48–49. **Solving pendulum equations** Use Exercise 47 and consider the following pendulum problems.

**a.** Solve the initial value problem and graph the solution.

**b.** Express the solution in amplitude-phase form.

**c.** Determine the period of the pendulum.

48. \( m = 1 \text{ kg}, \ \ell = 4.9 \text{ m}, \ \theta(0) = 0.25, \ \theta'(0) = 0 \)

49. \( m = 10 \text{ kg}, \ \ell = 3 \text{ m}, \ \theta(0) = 0.4, \ \theta'(0) = -0.2 \)

50. **Buoyancy as a restoring force** Imagine a cylinder of length \( L \) and cross-sectional area \( A \) floating partially submerged in a calm lake (see figure). Assume that the density of the water is \( \rho_w = 1000 \text{ kg/m}^3 \) and the density of the cylinder is \( \rho < 1000 \text{ kg/m}^3 \). Archimedes’ principle states that the cylinder experiences an upward buoyant force equal to the weight of the water displaced by the cylinder.

![Equilibrium](image)

**a.** As shown in the figure, assume that when floating at rest a fraction \( y/L \) of the cylinder is submerged. Note that the weight of the cylinder is \( mg = \rho ALg \) and the weight of the displaced water is \( y \rho_w Ag \). Conclude that the fraction of the cylinder that is submerged is the ratios of the densities; that is, \( \frac{y}{L} = \frac{\rho}{\rho_w} \).

**b.** Let \( y = 0 \) correspond to the level of the bottom of the cylinder at equilibrium. Now suppose that the cylinder is pushed down from its equilibrium position and released. Let \( y(t) \) be the position of the bottom of the cylinder \( t \) seconds after it is released, where \( y \) increases in the downward direction. Applying Newton’s second law \( my'' = F_{\text{ext}} \), explain why the buoyant force is \( F_{\text{ext}} = -\rho_w Ag \) (in addition to the buoyant force that maintains the equilibrium).

**c.** Conclude that the cylinder undergoes undamped oscillations governed by the equation
\[ y'' = -\omega_0^2 y, \text{ where } \omega_0^2 = \frac{\rho_w g}{\rho L} \]

**d.** What is the period of the oscillations? How does the period vary with the length of the cylinder? Is the period greater when \( \rho = \rho_w \) or when \( \rho < \rho_w \)? Explain your answer.

**e.** According to this model, what is the period of the oscillation when \( \rho = \rho_w \)? Describe this situation physically.

51–52. **Solving buoyancy equations** Use Exercise 50 and consider the following buoyant cylinders that oscillate with the given initial conditions.

**a.** Solve the initial value problem and graph the solution.

**b.** Express the solution in amplitude-phase form.

**c.** Determine the period of the oscillation.

51. \( L = 4.9 \text{ m}, \ \frac{\rho}{\rho_w} = \frac{1}{7}, \ y(0) = 0.25, \ y'(0) = 0 \)

52. \( L = 2 \text{ m}, \ \frac{\rho}{\rho_w} = 0.7, \ y(0) = 0.5, \ y'(0) = 0 \)

53. **Compartment models and drug metabolism** Compartment models are used to simulate manufacturing processes, ecological systems, and the assimilation of drugs by the body (pharmacokinetics). The figure shows a two-compartment model for a drug assimilation problem. Compartment 1 corresponds to the blood volume, which has an input of the drug at a specified rate \( f \). Compartment 2 cor-

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Chapter 16 • Second-Order Differential Equations

### 54–55. Solving compartment models
Use Exercise 53 and consider the following compartment models with the given initial conditions.

**a.** Solve the initial value problem for the mass of drug in compartment 1.

**b.** Determine the mass of drug in compartment 2.

**c.** Graph the mass of drug in both compartments and interpret the solutions.

**Problem 54.** $k_1 = 0.1, k_2 = 0.02, k_3 = 0.05, f(t) = 1, x_1(0) = x_2(0) = 0$ (infusion; constant input)

**Problem 55.** $k_1 = 0.1, k_2 = 0.02, k_3 = 0.05, f(t) = 0, x_1(0) = 10, x_2(0) = 0$ (injection at $t = 0$)

### Quick Check Answers

1. Damped, free oscillations; undamped, forced oscillations.

2. Period = $2\pi/4.5$, 4.5 oscillations per $2\pi$ time units, 4.5/2 oscillations per unit time.

3. The sound is loudest at the peaks of the long wave: $t = \pi/2, 3\pi/2, \ldots$

4. The value of the limit is 0; solutions decay in magnitude for all $A$ and $a$. $\blacksquare$

#### 16.5 Complex Forcing Functions

The oscillator equation with damping and external forcing is the most general equation we have considered, and it is used to model both mechanical and electrical oscillators. For this reason, we investigate this equation in more detail.

### The Transfer Function

The oscillator equation

$$my'' + cy' + ky = f(t),$$

models a time-invariant system because the characteristics of the system, described by the coefficients $m$, $c$, and $k$, are constant in time. Dividing through this equation by $m$, it can be written

$$y'' + by' + \omega_0^2 y = F(t),$$

where

$$b = \frac{c}{m}, \omega_0^2 = \frac{k}{m},$$

and $F(t) = \frac{f}{m}$.

All terms in the equation are accelerations. Recall that $b$ measures the damping in the system and $\omega_0 > 0$ is the natural frequency of the oscillator. We consider the case of passive systems, in which $b > 0$.

As already shown, the solution to this equation has two components: a transient solution that involves the initial conditions and decays in time, and a (particular) steady-state solution that is determined by the forcing function $F$. Our goal is to study the steady-state solution in the presence of periodic forcing functions. Therefore, we make two assumptions to focus the discussion.
First, we assume zero initial conditions \( y(0) = y'(0) = 0 \), which removes the transient solution. Second, we consider oscillatory forcing functions of the form

\[
F(t) = F_0 \cos(\omega t + \alpha),
\]

where the amplitude \( F_0 \), the forcing frequency \( \omega \), and the initial phase \( \alpha \) are specified.

For time-invariant linear systems, it is advantageous to work with complex-valued exponential forcing functions. Noting that \( F_0 \cos(\omega t + \alpha) = \Re\{F_0 e^{i(\omega t + \alpha)}\} \), we consider forcing functions of the form

\[
F(t) = F_0 e^{i(\omega t + \alpha)} = F_0 e^{i\alpha} e^{i\omega t} = fe^{i\omega t}.
\]

The constant \( f = F_0 e^{i\alpha} \) is the initial complex amplitude; its magnitude is

\[
|f| = |F_0 e^{i\alpha}| = |F_0||e^{i\alpha}| = |F_0|.
\]

You will see that using a forcing function of this form simplifies the work of finding solutions. To find a particular solution of the equation

\[
y'' + by' + \omega_0^2 y = fe^{i\omega t},
\]

we once again apply the method of undetermined coefficients and use the trial solution \( y_p = Ae^{i\omega t} \), where \( A \) is to be determined. Substituting \( y_p \) into the equation results in the following calculation:

\[
(Ae^{i\omega t})'' + b(Ae^{i\omega t})' + \omega_0^2 Ae^{i\omega t} = fe^{i\omega t} \quad \text{Substitute } y_p,
\]

\[
-A\omega^2 e^{i\omega t} + b\omega e^{i\omega t} + \omega_0^2 Ae^{i\omega t} = fe^{i\omega t} \quad \text{Differentiate: } i^2 = -1.
\]

\[
e^{i\omega t}(A(-\omega^2 + ib\omega + \omega_0^2)) = fe^{i\omega t} \quad \text{Collect terms.}
\]

The common factor \( e^{i\omega t} \) is nonzero and may be cancelled. Recalling that the goal is to determine \( A \), we find that

\[
A(-\omega^2 + ib\omega + \omega_0^2) = f \quad \text{Cancel } e^{i\omega t}.
\]

\[
A = \frac{f}{-\omega^2 + ib\omega + \omega_0^2} \quad \text{Solve for } A.
\]

Therefore, the particular solution to the equation becomes

\[
y_p = Ae^{i\omega t} = \frac{1}{-\omega^2 + ib\omega + \omega_0^2} fe^{i\omega t} = H(\omega) fe^{i\omega t}.
\]

The function

\[
H(\omega) = \frac{1}{-\omega^2 + ib\omega + \omega_0^2}
\]

that we have defined is called the transfer function for the time-invariant linear system. It depends on the damping constant \( b \), the natural frequency \( \omega_0 \), and the forcing frequency \( \omega \). We write \( H \) as a function of \( \omega \), because we hold the system parameters \( b \) and \( \omega_0 \) fixed and analyze the response of the system as the forcing frequency \( \omega \) varies.
As a complex quantity, we can write the transfer function in the polar form

\[ H(\omega) = g(\omega)e^{i\gamma(\omega)}, \]

where \( g(\omega) = |H(\omega)| \) is the gain function of the system and the phase angle \( \gamma(\omega) \) is the phase lag function. Note that the input function \( F \) and the output \( \gamma \) have the same frequency \( \omega \). The phase lag function \( \gamma(\omega) \) gives the phase of the output relative to the input, for all times.

We show next that for passive systems (\( b > 0 \)), the phase lag function is negative (\( \gamma(\omega) < 0 \)), which means the output lags the input (hence the terminology). For active systems (\( b < 0 \)), the phase lag is positive and the output leads the input.

A short calculation (Exercise 22) allows us to split \( H \) into its real and imaginary parts:

\[ H(\omega) = \frac{1}{-\omega^2 + ib\omega + \omega_0^2} = \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}, \]

\[ H(\omega) = \frac{b\omega}{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}. \]

We now find that the gain function is

\[ g(\omega) = |H(\omega)| = \sqrt{H_r(\omega)^2 + H_i(\omega)^2} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}. \]

Similarly, the phase lag function is given by

\[ \tan \gamma(\omega) = \frac{H_i(\omega)}{H_r(\omega)} = \frac{-b\omega}{\omega_0^2 - \omega^2}. \]

As shown shortly, the phase lag function has values in the interval \(-\pi < \gamma(\omega) \leq 0\).

**SUMMARY**  Transfer, Gain, and Phase Lag Functions

The oscillator equation \( y'' + by' + \omega_0^2y = F(t) \) has the transfer function

\[ H(\omega) = g(\omega)e^{i\gamma(\omega)} = \frac{1}{-\omega^2 + ib\omega + \omega_0^2}, \]

where the gain function is

\[ g(\omega) = |H(\omega)| = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2\omega^2}} \]

and the phase lag function is given by

\[ \tan \gamma(\omega) = \frac{-b\omega}{\omega_0^2 - \omega^2}. \]

The transfer function completely defines the relationship between the input and the output (or response) of the system. Conceptually, the transfer function “transfers” the input into the output, in the following way:

\[
\text{Input: } F(t) = f e^{i\omega t} \quad \text{Output: } y_p(t) = H(\omega) f e^{i\omega t},
\]

Both the input and output are complex functions. However, it is their real parts that give directly measurable quantities. Specifically, the real part of the input is

\[ \text{Re} \{ F(t) \} = F_0 \cos(\omega t + \alpha), \]
and the real part of the output is

\[
\text{Re}\{\gamma(t)\} = \text{Re}\left\{H(\omega)f e^{i\omega t}\right\} = \text{Re}\left\{g(\omega)e^{i\gamma(\omega)}F_0 e^{i\omega t}\right\} = g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega)).
\]

Substitute for \(g\),

\[
\text{Re}\{\gamma(t)\} = g(\omega)F_0 \cos(\omega t + \alpha).
\]

We see that the output amplitude \(g(\omega)F_0\) is \(g(\omega)\) multiplied by the input amplitude \(F_0\), while the output lags the input at all times (because \(\gamma(\omega) < 0\) when \(b > 0\)).

### SUMMARY Solution of Forced Oscillator Equations

The oscillator equation

\[
y'' + by' + \omega_0^2 y = \text{Re}\left\{f e^{i\omega t}\right\} = F_0 \cos(\omega t + \alpha)
\]

has the solution (output)

\[
\text{Re}\{\gamma(t)\} = g(\omega)F_0 \cos(\omega t + \alpha + \gamma(\omega))
\]

where \(g\) is the gain function and \(\gamma\) is the phase lag function.

### Properties of the Transfer Function

The transfer function—specifically its components, the gain and phase lag functions—are fundamental in the design of acoustical and electrical filters, noise canceling devices, and other instruments. So let’s look at them more closely.

The forcing frequency that produces the maximum response of the system is often of practical interest. In some cases (often mechanical), this frequency is to be avoided, whereas in other cases (often electrical) it is sought because it corresponds to the optimal performance of a device. It can be shown (Exercise 23) that the gain function

\[
g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2 \omega^2}}
\]

has a maximum value at \(\omega = \sqrt{\omega_0^2 - \frac{b^2}{2}}\), provided \(b < \sqrt{2}\omega_0\), which corresponds to weak damping. In the strong damping case \((b \geq \sqrt{2}\omega_0)\), the maximum value of \(g\) occurs at \(\omega = 0\).

The case shown in Figure 16.30a \((b = 0.5\) and \(\omega_0 = 1)\) corresponds to weak damping, and the gain function is maximized when \(\omega \approx 0.935\). In general, as the damping \(b\) approaches zero, the location of the peak in the gain curve occurs closer and closer to \(\omega_0\), the natural frequency of the system. Recall that in an undamped system, the condition \(\omega = \omega_0\) gives rise to resonance (unbounded growth of solutions). In a weakly damped system, maximum amplitudes occur when \(\omega = \omega_0\).

Let’s look at the phase lag function (Figure 16.30b)

\[
\tan \gamma(\omega) = -\frac{b\omega}{\omega_0^2 - \omega^2}.
\]

Notice that \(\gamma(0) = 0\), which means that if the forcing frequency is zero \((\omega = 0)\), the driving force and the output are constant, and there is no lag between the input and the output. When \(\omega\) is small and positive, \(\gamma(\omega)\) is small and negative, implying that the response
of the oscillator closely follows the variations in the driving force. As \( \omega \) increases, \( \gamma(\omega) \) decreases monotonically and as \( \omega \to \omega_0 \), we see that \( \tan \gamma(\omega) \to -\infty \), which implies that \( \gamma(\omega) \to -\frac{\pi}{2} \). As \( \omega \) continues to increase, \( \tan \gamma(\omega) \to 0 \), which means \( \gamma(\omega) \to -\pi \).

We see that the phase lag function always lies in the interval \( -\pi < \gamma(\omega) \leq 0 \). Summarized briefly, the greater the forcing frequency, the greater the phase lag between the input and output.

**EXAMPLE 1** Gain and phase lag Consider the oscillator equation \( y'' + by' + \omega_0^2 y = F_0 \cos \omega t \), where \( \omega_0 = 1 \), \( b = 0.5 \), and \( F_0 = 2 \) (the gain and phase lag functions are shown in Figure 16.30).

**a.** Find the gain and phase lag functions.

**b.** Assuming \( y(0) = y'(0) = 0 \), find and graph the solution of equation \( \text{Re}\{y_p\} \) with forcing frequencies \( \omega = 0.8 \) and \( \omega = 2 \).

**SOLUTION**

**a.** With the given values of \( \omega_0 \) and \( b \), the gain function is

\[
g(\omega) = \frac{1}{\sqrt{(1 - \omega^2)^2 + \frac{\omega^2}{4}}}.
\]

and the phase lag function is given by

\[
\tan \gamma(\omega) = -\frac{\omega/2}{1 - \omega^2}.
\]

**b.** Recall that the particular solution of the differential equation is

\[
\text{Re}\{y_p\} = g(\omega)F_0 \cos (\omega t + \gamma(\omega)).
\]

Letting \( \omega = 0.8 \), we find that \( g(0.8) = 1.86 \) and \( \gamma(0.8) \approx -0.84 \). Therefore, with \( F_0 = 2 \), the solution is

\[
\text{Re}\{y_p\} = 1.86 \cdot 2 \cos (0.8t - 0.84) = 3.72 \cos (0.8t - 0.84).
\]

The graphs of the forcing function \( F(t) = 2 \cos 0.8t \) and the real part of the solution are shown in Figure 16.31. We see that the amplitude of the forcing function is \( F_0 = 2 \) and the amplitude of the solution is nearly twice that amount, reflecting the amplification by a factor of \( g(0.8) \approx 1.86 \). Also note that the first local maximum of the
solution (output) curve occurs after the first maximum of the forcing function curve, reflecting the phase lag. To estimate the actual shift in the curves, we write
\[
\cos (0.8t - 0.84) = \cos \left( 0.8 \left( t - \frac{0.84}{0.8} \right) \right) = \cos (0.8(t - 1.05)).
\]
In this form, we see the shift (to the right) of the solution curve relative to the forcing function curve is approximately 1.05, consistent with Figure 16.31. In general, the shift in the curves is given by
\[
\frac{\gamma(\omega)}{\omega}.
\]

With the higher forcing frequency \( \omega = 2 \), we find that \( g(2) \approx 0.32 \) and \( \gamma(2) \approx -2.82 \). The solution is now
\[
\text{Re} \left\{ \gamma_p \right\} = 0.32 \cdot 2 \cdot \cos (2t - 2.82) = 0.64 \cos (2t - 2.82).
\]
The graphs of the forcing function and the real part of the solution are shown in Figure 16.32. In this case, the amplitude of the solution (output) is reduced by a factor of \( g(2) \approx 0.32 \). The first maximum of the solution curve again lags the first maximum of the forcing function curve. The shift in the curves is
\[
\frac{\gamma(\omega)}{\omega} = \frac{\gamma(2)}{2} \approx -1.41, \text{ consistent with Figure 16.32.}
\]

A different view of the gain function is shown in Figure 16.33. In this figure we fix \( \omega_0 = 1 \) and let \( b = 0.3, 1, \) and 3. For \( b \geq \sqrt{2\omega_0} \) (strong damping), the amplitude curve has a single local maximum at \( \omega = 0 \) with a value of 1. In this case, the values of the gain

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function are less than 1 in magnitude, meaning the amplitude of the response (output) is less than the amplitude of the forcing function (input). At \( b = \sqrt{2}\omega_0 \), the gain curves “flip” and for \( b < \sqrt{2}\omega_0 \) (weak damping), we see values of the gain that are greater than 1. In this weak damping case, it is possible to amplify the amplitude of the response provided the forcing frequency is chosen appropriately.

**EXAMPLE 2**  A low-pass filter  Consider the LCR circuit shown in Figure 16.34.

- **a.** Derive the differential equation for \( v_{\text{out}} \).
- **b.** Find the transfer function and compute the gain function.
- **c.** Give the relationship between \( v_{\text{in}} \) and \( v_{\text{out}} \).
- **d.** Find the relationship among \( L \), \( C \), and \( R \) such that the gain function is monotonically decreasing, for \( 0 \leq \omega < \infty \).
- **e.** Compute the gain function and compute the ratio \( \frac{v_{\text{out}}}{v_{\text{in}}} \) (as a function of \( \omega \)) when \( L = 0.5 \), \( C = 1/25 \), and \( R = 6 \). Interpret this result.

**SOLUTION**

- **a.** By Kirchhoff’s voltage law (see Section 16.4) we have
  \[ v_{\text{in}} - RI - L \frac{dI}{dt} - v_{\text{out}} = 0. \]
  The voltage drop across the capacitor is \( v_{\text{out}} = \frac{Q}{C} \). Differentiating both sides of this expression and using \( I = Q'(t) \) gives \( v'_{\text{out}} = \frac{Q'}{C} = \frac{I}{C} \) or \( I = CV'_{\text{out}} \). Substituting for \( I \) in the voltage law gives a second-order differential equation for \( v_{\text{out}} \):
  \[ v_{\text{in}} - RCv'_{\text{out}} - LV''_{\text{out}} - v_{\text{out}} = 0. \]
  Dividing through by \( LC \) and rearranging, the equation becomes
  \[ v''_{\text{out}} + R \frac{1}{L} v'_{\text{out}} + \frac{1}{LC} v_{\text{out}} = \frac{1}{LC} v_{\text{in}} = \frac{1}{LC} \frac{f e^{i\omega t}}{v_{\text{in}}}, \]
  where we have assumed that the input function is the complex exponential \( v_{\text{in}} = f e^{i\omega t} \).
b. To obtain the transfer function, we use the definition given previously with \( b = \frac{R}{L} \) and \( \omega_0^2 = \frac{1}{LC} \). The result is that

\[
H(\omega) = \frac{1}{-\omega^2 + i\omega \frac{R}{L} + \frac{1}{LC}} = \frac{1}{\left(\frac{1}{LC} - \omega^2 - i\frac{R}{L}\right)^2 + \left(\frac{R}{L}\right)^2 \omega^2}
\]

We compute the gain function

\[
g(\omega) = |H(\omega)| = \sqrt{H_r(\omega)^2 + H_i(\omega)^2}
\]

and find that

\[
g(\omega) = \frac{1}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R}{L}\right)^2 \omega^2}}
\]

c. The input to the system is \( v_{in} = f e^{i\theta} \) and the right side of the differential equation is \( \frac{1}{LC} v_{in} \). Therefore, the output of the system is

\[v_{out} = H(\omega) \frac{1}{LC} v_{in}.
\]

d. To analyze the behavior of the gain function, we locate the local extrema of \( g \). Observe that \( g \) has a local maximum at the same points that the denominator has a local minimum. Furthermore, to minimize the denominator, we can simply minimize the quantity

\[
h(\omega) = \left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R}{L}\right)^2 \omega^2
\]

under the square root in the denominator. To find the critical points of \( h \), we compute

\[
h'(\omega) = 2\left(\frac{1}{LC} - \omega^2\right)(-2\omega) + 2\left(\frac{R}{L}\right)^2 \omega \quad \text{Differentiate.}
\]

\[= -4\omega^2 + 2\left(\frac{R}{L}\right)^2 \omega \quad \text{Simplify.}
\]

We see that \( h'(0) = 0 \), so \( \omega = 0 \) is a critical point for all values of \( L, C \), and \( R \). Additional critical points, if they exist, satisfy

\[2\left(\omega^2 - \frac{1}{LC}\right) + \left(\frac{R}{L}\right)^2 = 0.
\]

After some simplification, this condition becomes

\[
\omega^2 = \frac{1}{LC} - \frac{R^2}{2L^2}.
\]

The critical point on the interval \( \omega > 0 \) is

\[\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{2L^2}}.
\]
provided \( \frac{1}{LC} - \frac{R^2}{2L^2} \geq 0 \) or \( R \leq \sqrt{\frac{2L}{C}} \). A bit more analysis shows that this critical point corresponds to a local minimum of \( h \) and therefore, a local maximum of \( g \). To summarize,

- if \( R \leq \sqrt{\frac{2L}{C}} \), then \( g \) has a local maximum for \( \omega > 0 \);
- if \( R > \sqrt{\frac{2L}{C}} \), then \( g \) has a local maximum at \( \omega = 0 \) and is monotonically decreasing for \( \omega \geq 0 \).

Notice that for all parameter values \( g(0) = LC \).

e. Substituting the given values of \( L, C, \) and \( R \), we see that \( R \leq \sqrt{\frac{2L}{C}} \), so we expect the gain function to have a single maximum at \( \omega = 0 \). Using these parameter values, the gain function is

\[
g(\omega) = \left| H(\omega) \right| = \frac{1}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + \left( \frac{R}{L} \omega \right)^2}} = \frac{1}{\sqrt{(50 - \omega^2)^2 + 144\omega^2}}
\]

To interpret the gain function, we compute the magnitude of the ratio of the output to the input. Using part (c), we have

\[
\left| \frac{v_{\text{out}}}{v_{\text{in}}} \right| = \left| \frac{H(\omega)}{v_{\text{in}}} \right| = \frac{g(\omega)}{LC} \quad \text{Substitute.}
\]

\[
= \frac{g(\omega)}{LC} \quad \text{Simplify: } \left| H(\omega) \right| = g(\omega).
\]

\[
= \frac{1}{LC \sqrt{(50 - \omega^2)^2 + 144\omega^2}} \quad \text{Substitute for } g.
\]

By graphing this ratio, we can see the effect of various forcing frequencies \( \omega \) on the input signal. The graph (Figure 16.35) indicates that all forcing frequencies produce an attenuation of the input signal. However, the low frequencies are attenuated the least, whereas frequencies with \( \omega > 10 \) are reduced in magnitude by a factor of at least 0.5. Because the low frequencies are allowed to “pass,” the circuit is called a low-pass filter.

> A similar analysis for a high-pass filter is given in Exercises 24–28.

![Figure 16.35](image.png)

Related Exercises 17–20
16.5 Complex Forcing Functions

Extensions
We have shown how the transfer function is used to find and interpret solutions of oscillator equations. The extensions of these ideas are also powerful. Let’s proceed in steps.

Suppose the forcing function consists of two components with two different frequencies \( \omega_1 \) and \( \omega_2 \); that is,

\[
F(t) = F_1 e^{j \omega_1 t} + F_2 e^{j \omega_2 t}.
\]

Because the differential equation is linear, we may superimpose the solutions corresponding to each component individually. The resulting solution is

\[
y = H(\omega_1)F_1 e^{j \omega_1 t} + H(\omega_2)F_2 e^{j \omega_2 t}.
\]

Recall that once \( b \) and \( \omega_0 \) are specified, we have a single transfer function, and it is easy to compute \( H(\omega_1) \) and \( H(\omega_2) \).

Now suppose that the forcing function is a sum of \( n \) components with different frequencies and amplitudes; that is,

\[
F(t) = \sum_{k=1}^{n} F_k e^{j \omega_k t}.
\]

Once again, we may superimpose solutions of individual equations and give the solution in the form

\[
y = \sum_{k=1}^{n} H(\omega_k)F_k e^{j \omega_k t}.
\]

As before, once the transfer function is determined, it is easy to compute the values of \( H(\omega_k) \), for \( k = 1, \ldots, n \).

We can now make a remarkable leap. Suppose the forcing function \( F \) is any smooth periodic function. Then it is possible to represent \( F \) in terms of a Fourier series of the form

\[
F(t) = \sum_{k=1}^{\infty} F_k e^{j \omega_k t}.
\]

Formally the method of solution outlined above can be carried out and the solution can also be expressed in terms of a Fourier series.

Let’s take one final step. If the forcing function is any function—not necessarily periodic—that meets fairly general conditions, it can be expressed in terms of a Fourier integral. Again the method of solution outlined above works and the solution can be expressed in terms of a Fourier integral. The simple idea of using a complex exponential forcing function proves to be incredibly powerful and leads to the solution of a wide range of oscillator problems.

SECTION 16.5 EXERCISES

Review Questions
1. Write the transfer function in terms of the gain and phase lag functions.
2. For the systems discussed in this section, explain the relationship between the input and the output in terms of the gain function.
3. For the systems discussed in this section, explain the relationship between the input and the output in terms of the phase lag function.
4. Use a graph to explain what it means for the output to lag the input.

Basic Skills
5–10. Gain and phase lag functions Consider the oscillator equation \( y'' + by' + \omega_0^2 y = F(t) \cos \omega t \).
   a. Write and graph the gain function and the phase lag function for following systems.
   b. Find the location of the local maximum of the gain function for \( \omega \approx 0 \). State whether the system has strong or weak damping.
5. \( b = 1, \omega_0 = 1 \)
6. \( b = 3, \omega_0 = 1 \)
7. \( b = 0.1, \omega_0 = 1 \)
8. \( b = 90, \omega_0 = 100 \)
9. \( b = 150, \omega_0 = 100 \)
10. \( b = 300, \omega_0 = 400 \)

### 11–16 Solutions to oscillator equations
Consider the oscillator equation \( y'' + b y' + \omega_0^2 y = F_0 \cos \omega t \). For the following parameter values, do the following.

a. Find the gain and phase lag functions (see Exercises 5–10).
b. Find the real part of the solution.
c. Graph the forcing function and the real part of the solution. Verify that the gain function and phase lag function are correct.

11. \( b = 1, \omega_0 = 1, F_0 = 2, \omega = 2 \)
12. \( b = 3, \omega_0 = 1, F_0 = 10, \omega = 1 \)
13. \( b = 0.1, \omega_0 = 1, F_0 = 5, \omega = 3 \)
14. \( b = 90, \omega_0 = 100, F_0 = 50, \omega = 50 \)
15. \( b = 150, \omega_0 = 100, F_0 = 80, \omega = 200 \)
16. \( b = 300, \omega_0 = 400, F_0 = 150, \omega = 200 \)

### 17–20. Analyzing circuit equations
Consider the circuit equation
\[
\frac{d^2 v_{\text{out}}}{dt^2} + \frac{R}{L} \frac{dv_{\text{out}}}{dt} + \frac{1}{LC} v_{\text{out}} = \frac{1}{LC} v_{\text{in}} = \frac{1}{LC} f e^{j\omega t}.
\]

a. For the given parameter values, compute the gain function.
b. Graph the gain function and find the location of its maximum value for \( \omega \geq 0 \).
c. On what interval is the gain function monotonically decreasing?
d. Compute and graph the ratio \( \frac{v_{\text{out}}}{v_{\text{in}}} \) and interpret the result.

17. \( L = 2, C = 1/444, R = 20 \)
18. \( L = 2, C = 1/444, R = 40 \)
19. \( L = 8, C = 1/120, R = 400 \)
20. \( L = 8, C = 1/100, R = 30 \)

### Further Explorations
21. Explain why or why not
Determine whether the following statements are true and give an example or a counterexample.

a. The transfer function for a forced oscillator equation is independent of the forcing function.
b. The transfer function for an undamped oscillator is a real-valued function.

22. Real and imaginary parts
Show that the transfer function discussed in this section can be expressed in terms of its real and imaginary parts in the form
\[
H(\omega) = \frac{1}{\omega^2 - \omega_0^2 + i\omega \omega_0} = \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + b^2 \omega^2} - \frac{b \omega_0}{(\omega_0^2 - \omega^2)^2 + b^2 \omega^2}.
\]

23. Maximum value of the gain function
The goal is to determine the location of the maximum value of the gain function
\[
g(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + b^2 \omega^2}}.
\]

a. Explain why the local maximum values of \( g \) for \( \omega \geq 0 \), occur at the same points as the local minimum values of the denominator.
b. Explain why the local maximum values of \( g \) for \( \omega \geq 0 \), occur at the same points as the local minimum values of \( h(\omega) = (\omega_0^2 - \omega^2)^2 + b^2 \omega^2 \).
c. Show that \( h'(\omega) = 2\omega(a^2 b^2 - 2a^2 + 2b^2) \).
d. Show that if \( b < \sqrt{2\omega} \), then \( g \) has a local maximum at \( \omega = \sqrt{\omega_0^2 - \frac{b^2}{2}} \). What is the maximum value of \( g \)?
e. Show that if \( b = \sqrt{2\omega} \), then \( g \) is monotonically decreasing for \( \omega \geq 0 \), and has a local maximum at \( \omega = 0 \).
f. Which case, part (d) or part (e), corresponds to strong damping? Explain.

24. A high-pass filter
Consider the LCR circuit shown in the figure.

[Diagram of LCR circuit with voltage source \( V_{\text{in}} \), inductor \( L \), capacitor \( C \), and resistor \( R \).]

a. Explain why Kirchhoff’s voltage law for the circuit is
\[
v_{\text{in}} - RI(t) - \frac{Q(t)}{C} - v_{\text{out}} = 0.
\]
b. Use the facts that \( i(t) = \frac{Q'}{L} \) and the voltage across the inductor is \( V_{\text{out}} = L\frac{Q'}{L} = LQ'' \) to show that the equation for the charge on the capacitor is
\[
Q'' + \frac{R}{L} Q' + \frac{1}{LC} Q = \frac{1}{L} \frac{V_{\text{in}}}{L} = \frac{1}{L} f e^{j\omega t}.
\]
c. Find the transfer function for the equation and compute the gain function.
d. Show that the solution of the differential equation is
\[
Q(t) = H(\omega) \frac{1}{L} f e^{j\omega t}.
\]
Then conclude that \( Q''(t) = -\omega^2 Q(t) \).
e. Now we must relate \( v_{\text{out}} \) to \( Q \). Use part (b) to show that
\[
v_{\text{out}} = -\omega^2 LQ = -\omega^2 L H(\omega) \frac{1}{L} e^{i\omega t} = -\omega^2 H(\omega) v_{\text{in}}.
\]

f. The quantity of interest is the ratio of the magnitudes of the input and the output. Show that
\[
\frac{|v_{\text{out}}|}{|v_{\text{in}}|} = \omega^2 |H(\omega)| = \omega^2 g(\omega) = \frac{\omega^2}{\sqrt{(\omega^2 - \frac{1}{LC})^2 + \frac{R^2}{L^2}}}
\]

g. Show that for \( R > \sqrt{\frac{2\pi}{C}} \) the ratio in part (f) is monotonically increasing for \( \omega > 0 \).

h. Under what conditions, if any, does the ratio in part (f) have a local extremum for \( \omega > 0 \)?

i. Explain why the circuit is called a high-pass filter.

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**CHAPTER 16 REVIEW EXERCISES**

1. **Explain why or why not** Determine whether the following statements are true and give an example or a counterexample.
   a. The equation \( y'' + 2y' - ty = 0 \) is second order, linear, and homogeneous.
   b. The equation \( y'' + 2y' - y^2 = 3 \) is second order, linear, and nonhomogeneous.
   c. To solve the equation \( t^2 y'' + 2ty' + 4y = 0 \), use the trial solution \( y = t^2 \) and find \( r \).
   d. To find the particular solution of the equation \( y'' - 2y' + 4y = e^t \), use the trial solution \( y_p = Ae^t \).
   e. The function \( y'' = 0 \) is always a solution of a second-order linear homogeneous equation.

2–9. **Solving homogeneous equations** Find the general solution of the following homogeneous equations.

2. \( y'' + y' - 6y = 0 \)
3. \( y'' - 2y' - 8y = 0 \)
4. \( y'' + 8y = 0 \)
5. \( y'' + 36y = 0 \)
6. \( y'' + 4y' + 13y = 0 \)
7. \( y'' + 4y' + 4y = 0 \)
8. \( t^2 y'' + 3ty' + 8y = 0 \), for \( t > 0 \)
9. \( y'' - 2y' + 5y = 0 \)

10–15. **Particular solutions** Find a particular solution of the following equations.

10. \( y'' - 3y' - 4y = 2e^{2t} \)
11. \( y'' + 25y = 3 \cos 2t \)
12. \( y'' - y' - 2y = 3t^2 + 10 \)
13. \( y'' + 5y' - 6y = t + 2e^{-t} \)

14. \( y'' - y' - 6y = 2e^{-2t} \)
15. \( y'' + 16y = 2 \cos 4t \)

16–21. **General solutions** Find the general solution of the following equations.

16. \( y'' - 6y' - 7y = 4e^t \)
17. \( y'' + 4y = 3 \sin 3t \)
18. \( y'' - 2y' + 2y = 1 + e^{-t} \)
19. \( y'' + 4y' + 5y = 2 \cos t \)
20. \( y'' + 4y' - 32y = 1 + 2t \)
21. \( y'' - y = 2e^t \)

22. **Forced undamped oscillations** A 4-kg block hangs on a spring with spring constant \( k = 16 \) N/m. The system is driven by an external force \( F_{\text{ext}} = 4 \cos \omega t \). Assume no initial displacement and velocity of the block, and no damping. Graph the solution in two cases: \( \omega = 2.2 \) and \( \omega = 4 \). In which case does the solution have beats?

23. **Free damped oscillations** A 0.2-kg block hangs on a spring with spring constant \( k = 20 \) N/m. Friction in the system is modeled with a damping coefficient of \( c \) kg/s. Solve an initial value problem with the following values of \( c \) and initial conditions \( y(0) = 1, y'(0) = 0 \). Graph each solution and state whether the problem exhibits underdamping, overdamping, or critical damping.
   a. \( c = 3 \) kg/s
   b. \( c = 4 \) kg/s
   c. \( c = 5 \) kg/s

24. **Forced damped oscillations** A 10-kg block hangs from a spring with spring constant \( k = 62.5 \) N/m. The block oscillates in a fluid that produces resistance with a damping constant \( c = 40 \) kg/s. The support of the spring vibrates and produces an external force...
Chapter 16 • Second-Order Differential Equations

of \( F_{\text{ext}} = 10 \cos 2t \). Assume the block is initially at rest at the equilibrium position and released with no velocity.

a. Find and graph the position of the block after it is released.
b. Identify the transient and steady state solutions, and graph them separately.
c. After approximately how many seconds does the transient solution become negligible?
d. Find the position function of the block using a transfer function and confirm that your solution is consistent with part (a).

Chapter 16 Guided Projects

Applications of the material in this chapter and related topics can be found in the following Guided Projects. For additional information, see the Preface.

- Oscillators

25. An LCR circuit An LCR circuit consists of a 10-ohm resistor, a 0.1-henry inductor, and a 1/240-farad capacitor connected in series to a variable voltage source with output \( E(t) = 200 \sin 60t \).

a. Find the current in the circuit, assuming that \( I(0) = Q(0) = 0 \).
b. Identify the transient and steady-state currents.
c. Graph the steady-state, transient, and total current, for \( t \geq 0 \).
d. Find the current using a transfer function and check for agreement with part (a).