1. A Banach space cannot have a countably infinite Hamel basis
   Let $X$ be an infinite-dimensional Banach space.
   (a) If $V \subset X$ is a finite-dimensional subspace, show that $V$ is closed.
   (b) Let $B \subset X$ be a maximal set of linearly independent vectors. It is a theorem of Linear Algebra that every element $x \in X$ can be uniquely represented as a finite linear combination
   \[
   x = \sum_{j=1}^{n} \alpha_j b_j
   \]
   for some nonnegative integer $n$, coefficients $\alpha_1, \ldots, \alpha_n$, and vectors $b_1, \ldots, b_n \in B$. Prove that $B$ is uncountable.

2. Compute the weak (distributional) derivatives of the function $f(x) = \sin x$ up to order 4.

3. The fundamental lemma of the Calculus of Variations
   Prove: If $F \in (C_c^\infty)'(\mathbb{R})$ satisfies $d/d_x F = 0$ in the sense of distributions, then $F$ is (represented by) a constant function, i.e., there exists a constant $c$ such that $F(\phi) = c \int \phi$ for all test functions $\phi \in C_c^\infty$.
   
   Hint: First consider the case where $\int \phi = 0$. (Write $\phi = d\Phi/dx$ for some test function $\Phi$.)

4. Fourier transform of $|x|^{-\lambda}$
   For $\lambda \in (0, n)$, consider the function defined on $\mathbb{R}^n \setminus \{0\}$ by $f_\lambda(x) = |x|^{-\lambda}$.
   (a) Verify that $f_\lambda$ defines a tempered distribution $F_\lambda \in S'$.
   (b) Suppose you know that its Fourier transform $\hat{F}_\lambda \in S'$ is also represented by a function (denoted by $\hat{f}_\lambda$). Use rotations and dilations to see that $\hat{f}_\lambda$ must have the form $\hat{f}_\lambda = Cf_{n-\lambda}$ for some constant $C = C_{n, \lambda}$.
   (c) Argue that the constant $C_{n, \lambda}$ is real. It turns out that the constant is also positive (see below; you are not asked to prove this). Conclude that the Coulomb energy
   \[
   \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \phi(x)|x - y|^{-1} \hat{\phi}(y) \, dx \, dy = C_{3, 1} \int_{\mathbb{R}^3} |k|^{-2} |\hat{\phi}(k)|^2 \, dk
   \]
   is real and positive for every $\phi \in S(\mathbb{R}^3)$.

   Remark: A proper computation of $\hat{F}_\lambda$ can be found in Lieb & Loss, Theorem 5.9.