1. **Nowhere monotone continuous functions**

   Prove that there exists a continuous function on the unit interval that is not monotone on any subinterval of positive length.

   *Hint:* Given a closed subinterval $I \subset [0, 1]$ of positive length, prove that the set
   
   $$A_I = \{ f \in C([0, 1]) : \text{if } |f| \text{ is monotone} \}$$

   is closed and contains no open balls, and then apply the Baire category theorem. You may use that $C([0, 1])$, the space of continuous function with the sup norm, is a Banach space, and that the piecewise linear functions form a dense subspace.

2. Let $X$ be a real Banach space, with dual space $X^*$.
   
   (a) *(Folland 5.25)*
   
   If $X^*$ is separable, prove that $X$ is separable.
   
   *Hint:* For each $f \in X^*$ there exists a unit vector $x \in X$ such that $|f(x)| \geq \frac{1}{2}||f||$. (Why?) Use this to construct a countable subset of $X$ that spans a dense subspace.

   (b) Last semester, you showed that $L^p(\mathbb{R}^d)$ is separable for $1 \leq p < \infty$ but not for $p = \infty$. Conclude that $L^1(\mathbb{R}^d)$ is a proper subspace of $(L^\infty(\mathbb{R}^d))^*$.

3. (a) Prove the **Hausdorff-Young inequality**

   $$||\hat{f}||_q \leq ||f||_p, \quad 1 \leq p \leq 2$$

   for $f \in L^p(\mathbb{R}^d)$ and a suitable value of $q$. Here, $\hat{f}$ is the Fourier transform of $f$.
   
   *Remark:* The inequality reduces to Parseval’s identity for $p = 2$, and fails for $p > 2$.

   (b) W. Beckner (1975) proved the sharp inequality

   $$||\hat{f}||_q \leq C(p, q, d)||f||_p, \quad 1 \leq p \leq 2,$$

   where $C(p, q, d) \leq 1$ are explicit constants, and showed that equality occurs when $f$ is a Gaussian. Taking Beckner’s theorem for granted, compute the values of these constants.
4. **Weak = strong convergence in \(\ell^1\) (Brézis Problem 8)**

Let \(\ell^1\) denote the space of summable sequences \(x = (x_1, x_2, \ldots)\), with norm \(\|x\|_1 = \sum |x_i|\). Its dual is \(\ell^\infty\), the space of bounded sequences \(f = (f_1, f_2, \ldots)\) with norm \(\|f\|_\infty = \sup |f_i|\).

We essentially showed in class that the weak-* topology on the closed unit ball \(B^* \subset \ell^\infty\) is generated by the metric

\[
d(f, g) = \sum_{i=1}^{\infty} 2^{-i}|f_i - g_i|.
\]

(Note that the sequences \(e_n\) given by \(e_n^i = 1\) for \(i = n\) and 0 otherwise form a Schauder basis of \(\ell^1\)). By Alaoglu’s theorem, \(B^*\) is compact, and in particular complete.

(a) Let \((x_n)\) be a sequence in \(\ell^1\) that converges weakly to zero \((x_n \rightharpoonup 0)\), and let \(\varepsilon > 0\) be given. Prove that there exists an integer \(N\) such that

\[
F_N = \{ f \in B^* \mid |f(x_n)| \leq \varepsilon \ \forall \ n \geq N \}
\]

contains an open ball \(U_{\rho, g} = \{ f \in B^* \mid d(f, g) < \rho \} \) for some \(g \in B^*\) and \(\rho > 0\).

(b) Deduce from (a) that

\[
d(f, 0) < \rho \implies |f(x_n)| \leq 2\varepsilon \ \forall \ n \geq N.
\]

(c) Fix an integer \(k\) such that \(2^{-k} < \rho\). Prove that

\[
\|x_n\|_1 \leq \sum_{i=1}^{k} |x_n^i| + 2\varepsilon \ \forall \ n \geq N.
\]

Then take \(n \to \infty\). **Hint:** Consider \(f \in B^*\) with \(f^i = 0\) for \(i = 1, \ldots, k\).

(d) Conclude that for arbitrary sequences \((x_n)\) in \(\ell^1\),

\[
x_n \rightharpoonup a \text{ weakly in } \ell^1 \implies \lim \|x_n - a\|_1 = 0.
\]

**Remark:** You’ve shown that weak convergence in \(\ell^1\) is equivalent to convergence in norm (even though the weak topology is strictly coarser than the norm topology)!