1. **Nowhere monotone continuous functions**

   Prove that there exists a continuous function on the unit interval that is not monotone on any subinterval of positive length.

   **Hint:** Given a closed subinterval $I \subset [0, 1]$ of positive length, prove that the set
   
   $$A_I = \{ f \in C([0, 1]) : \text{$f|_I$ is monotone} \}$$

   is closed and contains no open balls, and then apply the Baire category theorem. You may use that $C([0, 1])$, the space of continuous function with the sup norm, is a Banach space, and that the piecewise linear functions form a dense subspace.

2. Let $X$ be a real Banach space, with dual space $X^*$.

   (a) *(Folland 5.25)*

   If $X^*$ is separable, prove that $X$ is separable.

   **Hint:** For each $f \in X^*$ there exists a unit vector $x \in X$ such that $\|f(x)\| \geq \frac{1}{2}\|f\|$. (Why?) Use this to construct a countable subset of $X$ that spans a dense subspace.

   (b) *(Folland 6.12)*

   Show that $L^p(\mathbb{R}^d)$ is separable for $1 \leq p < \infty$ but not for $p = \infty$.

   (c) Conclude that $L^1(\mathbb{R}^d)$ is a proper subspace of $(L^\infty(\mathbb{R}^d))^*$.

3. (a) Prove the **Hausdorff-Young inequality**

   $$\|\hat{f}\|_q \leq \|f\|_p, \quad 1 \leq p \leq 2$$

   for $f \in L^p(\mathbb{R}^d)$ and a suitable value of $q$. Here, $\hat{f}$ is the Fourier transform of $f$.

   **Remark:** The inequality fails for $p > 2$.

   (b) W. Beckner (1975) proved the sharp inequality

   $$\|\hat{f}\|_q \leq C(p, q, d)\|f\|_p, \quad 1 \leq p \leq 2,$$

   where $C(p, q, d) \leq 1$ are explicit constants, and showed that equality occurs when $f$ is a Gaussian. Taking Beckner’s theorem for granted, compute the values of these constants.
4. **Weak = strong convergence in** $\ell^1$ (*Brézis Problem 8*)

Let $\ell^1$ denote the space of summable sequences $x = (x^1, x^2, \ldots)$, with norm $||x||_1 = \sum |x^i|$. Its dual is the space $\ell^\infty$ of bounded sequences $f = (f^1, f^2, \ldots)$ with norm $||f||_\infty = \sup |f^i|$. We showed in class that the weak-* topology on the closed unit ball $B^* \subset \ell^\infty$ is generated by the metric

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-i} |f^i - g^i|.$$ 

By Alaoglu’s theorem, $B^*$ is compact, and in particular complete.

(a) Let $(x_n)$ be a sequence in $\ell^1$ that converges weakly to zero ($x_n \rightharpoonup 0$), and let $\varepsilon > 0$ be given. Prove that there exists an integer $N$ such that

$$F_N = \{ f \in B^* | |f(x_n)| \leq \varepsilon \forall n \geq N \}$$

contains a an open ball in $B^*$.

(b) Deduce from (a) that there exists $\rho > 0$ such that

$$||f||_\infty < \rho \quad \Rightarrow \quad |f(x_n)| \leq 2\varepsilon \forall n \geq N.$$ 

Therefore $||x_n||_1 \leq 2\varepsilon / \rho$ for all $n \geq N$.

**Hint:** Note that $d(f, g) \leq ||f - g||_\infty$ and apply the triangle inequality.

(c) Conclude that for arbitrary sequences $(x_n)$ in $\ell^1$, 

$$x_n \rightharpoonup a \text{ weakly in } \ell^1 \quad \Rightarrow \quad \lim ||x_n - a||_1 = 0.$$ 

**Remark:** You’ve shown that weak convergence in $\ell^1$ is equivalent to convergence in norm (even though the weak topology is strictly coarser than the norm topology)!