1. Let $X$ be a normed vector space.
   (a) Find the Legendre transform of $\phi(x) = \|x\|$. 
   (b) If $C \subset X$ is open, convex, and contains the origin, find the Legendre transform of its Minkowski functional
       
       $p(x) = \inf\{t > 0 \mid t^{-1}x \in C\}$. 
       
       (Is it clear that $p$ is convex, lower semicontinuous, and proper?)

2. (Folland 6.16) 
   If $0 < p < 1$, show that the formula $\rho(f, g) = \int |f - g|^p$ defines a metric on $L^p$ that makes $L^p$ into a complete topological vector space. (You need to verify the triangle inequality, the continuity of translation and dilation, and completeness.)

3. (Folland 5.63) 
   Let $X$ be a normed vector space. We say that a sequence $(x_n)_{n \geq 1}$ converges weakly to a limit $a$ in $X$ (and write $x_n \rightharpoonup a$), if
   
   \[ \lim_{n \to \infty} \phi(x_n) = \phi(a) \]
   
   for all $\phi \in X^*$. 
   
   Let $H$ be an infinite-dimensional Hilbert space. In that case, $x_n \rightharpoonup a$ means that
   
   \[ \lim_{n \to \infty} \langle x_n, v \rangle = \langle a, v \rangle \]
   
   for all $v \in H$. Prove that ...
   
   (a) ... every orthonormal sequence in $H$ converges weakly to zero; 
   
   (b) ... the unit sphere $S = \{x \in H : \|x\| = 1\}$ is weakly dense in the closed unit ball $B = \{x \in H : \|x\| \leq 1\}$, i.e, every $x \in B$ is the weak limit of a sequence of unit vectors.
4. (a) Solve the heat equation

\[ \partial_t u = \Delta u, \quad (x \in \mathbb{R}^n, t > 0) \]

with initial values

\[ u(x, 0) = f(x), \quad (x \in \mathbb{R}^n) \]

by deriving a differential equation for the Fourier transform \( \hat{u}(k,t) = \int e^{-2\pi ik \cdot x} u(x,t) \, dx \).

Here, \( \Delta u = \sum_j \partial^2_{x_j} u \) is the Laplacian, and \( f \) lies in the Schwartz space \( \mathcal{S} \).

Don’t forget to transform back ...

(b) You have obtained a formula

\[ u(x,t) = \int_{\mathbb{R}^n} K_t(x,y)f(y) \, dy. \]

Given an integrable function \( f \) on \( \mathbb{R}^d \), prove that this integral defines a function \( u \) that is smooth for \( t > 0 \) and satisfies the heat equation. Moreover, \( u(\cdot, t) \) converges to \( f \) in \( L^1 \).

(c) Prove that \( \lim_{t \to 0} u(x,t) = f(x) \) for every \( x \in \mathbb{R}^d \) where \( f \) is continuous.

Remark: In fact, \( u(x,t) \to f(x) \) for all \( x \) in the Lebesgue set of \( f \).

(You are not asked to prove this.)