1. **Riesz-Sobolev ⇒ Brunn-Minkowski**

We’ve proved in class that

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) g(x-y) h(x) \, dy \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f^*(y) g^*(x-y) h^*(x) \, dy \, dx
\]

holds for any three nonnegative functions \(f, g, \text{ and } h\) that vanish at infinity. Use this to give an alternate proof of the Brunn-Minkowski inequality

\[
m(A + B)^{\frac{1}{d}} \geq m(A)^{\frac{1}{d}} + m(B)^{\frac{1}{d}}
\]

for any pair of sets \(A, B \subset \mathbb{R}^d\) of finite positive measure.

**Hint:** Consider \(f = \mathcal{X}_A, g = \mathcal{X}_B, \text{ and } h = \mathcal{X}_{A+B}\).

2. **Symmetric decreasing rearrangement defines an \(L^p\)-contraction**

(a) Let \(f, g\) be nonnegative functions in \(L^p(\mathbb{R}^d)\), and let \(f^*, g^*\) be their symmetric decreasing rearrangements. If \(1 < p < \infty\), show that

\[
||f - g||_p^p = p(p-1) \int_0^\infty \int_0^t m(\{f > t\} \triangle \{g > s\}) + m(\{g > t\} \triangle \{f > s\}) \, |t-s|^{p-2} \, ds \, dt.
\]

(b) Conclude that \(||f^* - g^*||_p \leq ||f - g||_p\).

(c) Moreover,

\[
||f - g||_p^p \leq p \int_0^\infty m(\{f > t\} \triangle \{g > t\}) \, t^{p-1} \, dt.
\]

**Remark:** Inequality (b) extends to \(p = 1\) and \(p = \infty\); (c) is an identity for \(p = 1\).

3. **HLS ⇒ weak Young inequality**

For measurable functions on \(\mathbb{R}^d\), the functional

\[
||f||_{p,\text{weak}} = \sup_{t>0} \left\{ t \cdot m(\{|f(x)| > t\})^{\frac{1}{p}} \right\}
\]

is called the **weak \(L^p\)-norm**. Define the **weak \(L^p\)-space** by

\[
L^p_{\text{weak}} = \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ measurable} \mid ||f||_{p,\text{weak}} < \infty \right\} / \text{a.e.}
\]
The name of $|| \cdot ||_{p, \text{weak}}$ is a bit misleading, as the functional fails to satisfy the triangle inequality. To remedy this, one can construct a true norm that makes $L^p_{\text{weak}}$ into a Banach space, and which differs from $|| \cdot ||_{p, \text{weak}}$ at most by a multiplicative constant.

(a) Show that $L^p \subset L^p_{\text{weak}}$.

(b) Use the HLS inequality to show that for $p, q, r > 1$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, there is a constant $C = C(p, q, r, d)$ such that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y) g(x - y) h(x) \, dy \, dx \leq C ||f||_p ||g||_{q, \text{weak}} ||h||_r$$

for all $f, g, h$ such that the right hand side is finite. 

Hint: Apply the Riesz-Sobolev inequality.

4. HLS ($\lambda = d - 2$) $\Leftrightarrow$ Sobolev ($p = 2$) (Lieb & Loss Section 8.3)

On the Hilbert space $L^2(\mathbb{R}^d)$, consider the Legendre transform

$$\phi^*(g) = \sup_f \{ \langle f, g \rangle - \phi(f) \}.$$ 

(a) For $1 < p < \infty$, find the Legendre transform of

$$\phi_1(f) = \begin{cases} ||f||^2_p, & \text{if } f \in L^p \cap L^2 \\ +\infty, & \text{otherwise} \end{cases}.$$ 

Remark: The functional $\phi_1$ is clearly convex and proper. Accept without proof that it is also lower semicontinuous (and likewise for $\phi_2$ below).

(b) Also find the Legendre transform of

$$\phi_2(f) = \begin{cases} ||\nabla f||^2_2, & \text{if } f \in W^{1,2}, \\ +\infty, & \text{otherwise} \end{cases}.$$ 

Hint: Take $f \in S$ and apply the Fourier transform. Then use the result of Problem 4 on Assignment 7.

(c) Check that the Legendre transform is order-reversing.

$$\phi_1 \leq \phi_2 \Leftrightarrow \phi_1^* \geq \phi_2^*.$$ 

Remark: You have established that the Sobolev inequality

$$||\nabla f||^2_2 \geq S||f||^2_p,$$

is equivalent to the HLS inequality

$$\int \int \frac{g(x)g(y)}{|x - y|^{d-2}} \, dx \, dy \leq C ||g||^2_q,$$

via the Legendre transform. Note that $r$ and $q$ are Hölder dual exponents.