1. (a) Let \((u_n)\) be a sequence in a separable Hilbert space \(H\). What does it mean for \((u_n)\) to be an orthonormal basis? Please give three equivalent conditions.

(b) \((The\ Haar\ basis)\)
For \(n \geq 0\) and \(x \in [0, 1]\), let
\[
v_n(x) = 1 - 2X_n,
\]
where \(X_n\) is the \(n\)-th digit in the binary expansion of \(x\) (i.e. \(v_n(x) = 1\) when this digit is zero, and \(-1\) otherwise. If \(x\) has several expansions, use the one that terminates in 0). Prove that \((v_n)\) is an orthonormal basis for \(L^2([0, 1])\).

2. Let \((u_n)_{n \geq 1}\) be an orthonormal basis for a Hilbert space \(H\), and consider the linear transformation \(T: H \to H\) given by \(Tu_n = u_{n+1}\).

(a) Show that \(T\) is compact, and compute its norm.

(b) Show that \(T\) has no eigenvalues. Why does this not contradict the Spectral Theorem?

3. State the (Banach)-Alaoglu theorem (on a general Banach space). What does it imply about bounded sequences in \(L^p(\mathbb{R})\)? How? (Your answer will depend on the value of \(p\).)

4. Show that
\[
Tf(x) = \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x+y} \, dy
\]
defines a bounded linear operator on \(L^2(0, \infty)\).

Hint: Estimate \(\langle Tf, f \rangle\), using
\[
|f(x)\hat{f}(y)| \leq \frac{1}{2} \left( \left( \frac{x}{y} \right)^{\frac{1}{2}} |f(x)|^2 + \left( \frac{y}{x} \right)^{\frac{1}{2}} |f(y)|^2 \right). \]