1. **Hilbert-Schmidt operators**
   Let $K(x, y)$ be a complex-valued function in $L^2(\mathbb{R}^2)$, and set
   
   $$Tf(x) = \int_{\mathbb{R}} K(x, y) f(y) \, dy.$$ 

   (a) Show that $f \mapsto Tf$ defines a bounded linear operator on $L^2(\mathbb{R})$.
   (b) Find a formula for its adjoint, $T^*$.

2. **Local properties of convex functions (Folland 3.42abc)**
   Let $F$ be a convex function on an interval $(a, b)$, i.e.,
   
   $$F\left( (1 - \lambda)x + \lambda y \right) \leq (1 - \lambda)F(x) + \lambda F(y)$$
   
   for all $x, y \in (a, b)$ and all $\lambda \in (0, 1)$.
   (a) Prove that $F$ is absolutely continuous on compact subintervals, and that $F'$ is increasing.
   **Hint:** Argue that the difference quotient
   
   $$Q(x, y) = \frac{F(y) - F(x)}{y - x}, \quad (x < y)$$
   
   is increasing in both $x$ and $y$, and consider the one-sided derivatives $D_+ F$ and $D_- F$.
   (b) Conclude that for every $a$ there exists a constant $\beta$ such that
   
   $$F(x) \geq F(a) + \beta (x - a) \quad \text{for all } x \in \mathbb{R}.$$ 

   **Remark:** The line $\ell(x) = F(a) + \beta (x - a)$ is called a *support line* for $F$ at $a$. How does $\beta$ relate to $D_\pm F(a)$?
3. (a) Jensen’s inequality (Folland 3.42d)
   Let \( \mu \) be a probability measure on \( X \), and let \( F \) be a convex function. Then, for every integrable function \( f \) on \( X \),
   \[
   F \left( \int f \, d\mu \right) \leq \int (F \circ f) \, d\mu .
   \]
   (In particular, the right hand side is well-defined, though it may take the value \( +\infty \).)
   
   \textit{Hint:} Consider the support line of \( F \) at \( \bar{f} = \int f \, d\mu \).

(b) Hölder’s inequality
   Let \( f \in L^p(d\mu) \) and \( g \in L^q(d\mu) \), where \( \mu \) is an arbitrary measure and \( 1 < p, q < \infty \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). Use Jensen’s inequality to prove that their product \( fg \) is integrable, and
   \[
   \left| \int fg \right| \leq ||f||_p \cdot ||g||_q .
   \]
   
   \textit{Hint:} Consider the measure \( |g(x)|^q \, d\mu \), suitably normalized.

4. Fix \( p \in [1, \infty) \), and let \( f \) be a nonnegative \( p \)-integrable function on \( \mathbb{R}^n \).
   (a) Show that
   \[
   ||f||_p^p = \int_0^\infty \mu(\{x : f(x) > t\}) pt^{p-1} \, dt .
   \]

   (b) The \textit{symmetric decreasing rearrangement} of \( f \) is defined by
   \[
   f^*(x) = \int_0^\infty \chi_{|x|<r(t)} \, dt ,
   \]
   where \( r(t) \) is the radius of a ball that has the same measure as the level set \( \{x : f(x) > t\} \).
   Show that \( ||f^*||_p = ||f||_p \).