MAT 1001 / 458 : Real Analysis II
Assignment 4, due February 5, 2014

1. **Convolution operators on \( L^2([0, 2\pi]) \)**

   Let \( \mathcal{H} \) be the Hilbert space of \( 2\pi \)-periodic square integrable functions (with the usual identification of functions that agree a.e.) and with inner product \( \langle f, g \rangle = \int_0^{2\pi} f(x) \bar{g}(x) \, dx \). Fix a function \( h \in \mathcal{H} \), and consider the linear operator \( T : f \mapsto f * h \).

   (a) Verify that the functions \( u_n(x) = e^{inx}, \quad n \in \mathbb{Z} \) are eigenfunctions for \( T \). What are the corresponding eigenvalues \( \lambda_n \)?

   (b) Show that \( T \) is compact. Under what conditions on \( h \) is it self-adjoint?

2. **(Stein & Shakarchi, Exercise 4.32)**

   Prove that the operator \( T : L^2([0, 1]) \to L^2([0, 1]) \) defined by \((Tf)(x) = xf(x)\) is bounded and self-adjoint, but not compact. Moreover, \( T \) has no eigenvectors.

3. **Von Neumann’s alternating projection theorem**

   Let \( P_1 \) and \( P_2 \) be orthogonal projections onto closed subspaces \( V_1 \) and \( V_2 \) of a Hilbert space \( \mathcal{H} \), respectively, and let \( P \) be the orthogonal projection onto the intersection \( V = V_1 \cap V_2 \). Convince yourself that \( P_1 P_2 x = x \), if and only if \( x \in V \) (and the same holds for \( P_2 P_1 \)).

   You will show that \((P_1 P_2)^n x\) converges to \( Px \) for all \( x \in \mathcal{H} \). (Please make a sketch!)

   (a) Prove that \( \| x - P_1 P_2 x \|^2 \leq 2(\| x \|^2 - \| P_1 P_2 x \|^2) \) for all \( x \in \mathcal{H} \).

   (b) **Kakutani’s lemma**

   Given \( x \), set \( x_n = (P_1 P_2)^n x \), so that \( x_{n+1} = P_1 P_2 x_n \). Then \( \lim \| x_n - x_{n+1} \| = 0 \).

   (c) Conclude that \( \lim \| x_n - P x \| = 0 \). (Hint: Apply Problem 4.2(a) to \( R(I - P_1 P_2) \).)

4. **(Folland 6.6)**

   Let \((X, \mu)\) be a measure space. If \( f \) is a measurable complex-valued function on \( X \), set

   \[
   \|f\|_p := \begin{cases} 
   \left( \int |f|^p \, d\mu \right)^{1/p}, & 1 \leq p < \infty, \\
   \inf \{t > 0 \mid f(x) \leq t \text{ a.e.} \}, & p = \infty.
   \end{cases}
   \]

   Let \( L^p(X, d\mu) \) denote the space of functions with \( \|f\|_p < \infty \) (identifying functions that agree a.e.). We will show next week that \( L^p(X, d\mu) \) is a Banach space with norm \( \| \cdot \|_p \).

   Suppose \( 0 < p_0 < p_1 \leq \infty \). Find examples of functions on \( \mathbb{R}^+ \) (with Lebesgue measure) such that \( f \in L^p \) if and only if...

   (a) \( p_0 < p < p_1 \);

   (b) \( p_0 \leq p \leq p_1 \);

   (c) \( p = p_0 \).

   **Hint:** Try functions of the form \( f(x) = x^{-\alpha} |\log x|^\beta \).