MAT 1001 / 458 : Real Analysis II
Assignment 3, due January 29, 2014

Let $\mathcal{H}$ be a Hilbert space.

1. (Stein & Shakarchi, Exercise 4.7.9)

Let $\mathcal{H}_1 = L^2([-\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with inner product

\[ \langle F, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) \overline{G(e^{i\theta})} \, d\theta. \]

Let $\mathcal{H}_2$ be the space $L^2(\mathbb{R})$ with the usual inner product.

(a) Using the mapping $x \mapsto \frac{i - x}{i + x}$ of $\mathbb{R}$ to the unit circle, show that the corresponding transformation $U: F \mapsto f$, with

\[ f(x) = \frac{1}{\sqrt{\pi(i+x)}} F\left(\frac{i-x}{i+x}\right) \]

defines a unitary mapping of $\mathcal{H}_1$ to $\mathcal{H}_2$. What is its inverse?

(Hint: Change variables $x = \tan \frac{\theta}{2}$.)

(b) Conclude that

\[ \left\{ \frac{1}{\sqrt{\pi(i+x)}} \left(\frac{i-x}{i+x}\right)^n \right\}_{n \in \mathbb{Z}} \]

is an orthonormal basis of $L^2(\mathbb{R})$.

2. The adjoint of a linear transformation $T: \mathcal{H} \to \mathcal{H}$ is defined by the property that

\[ \langle Tx, y \rangle = \langle x, T^* y \rangle \quad \text{for all } x, y \in \mathcal{H}. \]

Clearly, $(T^*)^* = T$.

(a) (Folland 5.57)

Show that the ranges and nullspaces of $T$ and $T^*$ are related by

\[ \mathcal{R}(T)^\perp = \mathcal{N}(T^*), \quad \mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}. \]

Remark: In particular, if $T$ is self-adjoint ($T^* = T$), then $\mathcal{N}(T)$ and $\overline{\mathcal{R}(T)}$ are complementary orthogonal subspaces.

(b) (Folland 5.58)

Let $P$ be the orthogonal projection onto a closed subspace $V \subset \mathcal{H}$, i.e., for each point $x \in \mathcal{H}$, its image $Px$ is the point of $V$ closest to $x$. Verify that $P$ is self-adjoint, and that $P^2 = P$.

Conversely, if $P$ is self-adjoint and $P^2 = P$, show that $V = \mathcal{R}(P)$ is closed and $P$ is the orthogonal projection onto $V$. 

1
3. (Stein & Shakarchi, Exercise 4.7.13)
Let $P_1$ and $P_2$ be a pair of orthogonal projections onto closed subspaces $V_1$ and $V_2$ of $H$, respectively, and let $P$ be the orthogonal projection onto the intersection $V = V_1 \cap V_2$. Prove that $P_1P_2$ is an orthogonal projection, if and only if $P_1P_2 = P_2P_1$, i.e., the two projections commute. In that case, $(P_1P_2)^n = P_1P_2 = P$.

*Hint*: What can you say about $||P_1P_2x||$?

4. (Folland 5.63)

We say that a sequence $(x_n)_{n \geq 1}$ **converges weakly** to a limit $a$ in $H$ (and write $x_n \rightharpoonup a$), if

$$\lim \langle x_n, v \rangle = \langle a, v \rangle$$

for all $v \in H$. Assume that $H$ is infinite-dimensional. Prove that ...

(a) every orthonormal sequence in $H$ converges weakly to zero;

(b) the unit sphere $S = \{x \in H : ||x|| = 1\}$ is weakly dense in the closed unit ball $\overline{B} = \{x \in H : ||x|| \leq 1\}$, i.e., every $x \in \overline{B}$ is the weak limit of a sequence in $S$. 
