1. **Polarization identity (Folland 5.55)**
   (a) Show that for every \( x, y \in \mathcal{H} \),
   \[
   \langle x, y \rangle = \frac{1}{4} \left( \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 \right).
   \]
   (Remark: Completeness is not needed here.)
   (b) If a linear map \( L : \mathcal{H}_1 \to \mathcal{H}_2 \) between two Hilbert spaces is isometric and surjective, then it is **unitary**, i.e., \( L \) is invertible and \( \langle Lx, Ly \rangle = \langle x, y \rangle \) for all \( x, y \in \mathcal{H}_1 \).

2. **Orthogonal polynomials**
   (a) Let \( \mu \) be a measure on an interval such that \( L^2(d\mu) \) is infinite-dimensional and contains the space of polynomials as a dense subspace. Describe how to construct an orthogonal basis \( (p_n)_{n \geq 0} \) for \( L^2(d\mu) \) where each \( p_n \) is a polynomial of degree exactly \( n \).
   (b) **Three-term recurrence relation**
   Show that there exist real-valued sequences \( (a_n), (b_n), (c_n) \) such that
   \[
   p_{n+1} = (a_n x + b_n) p_n + c_n p_{n-1} \quad \text{for} \quad n \geq 1.
   \]
   **Hint:** Argue that that \( \langle xp_n, p_m \rangle = 0 \) if \( m + 1 < n \).

3. **Example of a non-separable Hilbert space (Stein & Shakarchi, Problem 4.2)**
   Consider the collection of exponential functions \( f_\lambda(x) = e^{i\lambda x} \) on the real line, where \( \lambda \) ranges over \( \mathbb{R} \). Let \( \mathcal{H}_0 \) denote the space of finite linear combinations of these exponentials. For \( f, g \in \mathcal{H}_0 \), define an inner product by
   \[
   \langle f, g \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x)\bar{g}(x) \, dx.
   \]
   (a) Show that this limit exists, and compute its value for \( f = \sum_{j=1}^{n} \alpha_j f_{\lambda_j}, \ g = \sum_{k=1}^{m} \beta_k f_{\mu_k} \).
   (b) Show that the corresponding norm, given by \( \|f\| = \sqrt{\langle f, f \rangle} \), satisfies
   \[
   \|f\| \leq \sup_{x} |f(x)|.
   \]
   (c) Let \( \mathcal{H} \) be the completion of \( \mathcal{H}_0 \) with respect to \( \| \cdot \| \). Prove that \( \mathcal{H} \) is not separable.
   **Hint:** The functions \( \{f_\lambda\}_{\lambda \in \mathbb{R}} \) are orthonormal.
4. A Banach space does not have a countably infinite Hamel basis

Let $X$ be an infinite-dimensional Banach space.

(a) If $V \subset X$ is a finite-dimensional subspace, show that $V$ is closed.
(b) Let $B \subset X$ be a maximal set of linearly independent vectors. Prove that $B$ is uncountable.

Remark: It is a theorem of Linear Algebra that $B$ spans $X$. More precisely, every element $x \in X$ can be uniquely represented as a finite linear combination

$$x = \sum_{j=1}^{n} \alpha_j b_j$$

for some nonnegative integer $n$, non-zero vectors $b_1, \ldots, b_n \in B$, and non-zero coefficients $\alpha_1, \ldots, \alpha_n$. Such a set is called a (Hamel) basis. The existence of a basis is guaranteed by Zorn’s lemma.