1. Find the values of (a) \( \sum_{k=1}^{\infty} \frac{1}{k^2} \), (b) \( \sum_{k=1}^{\infty} \frac{1}{k^4} \), (c) \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \).

*Hint:* Use the Fourier series of the function \( f(x) = x \) for \( x \in [-\pi, \pi] \).

2. **The Poisson summation formula**
   (a) Let \( f \in S(\mathbb{R}) \), and let \( \hat{f} \) be its Fourier (integral) transform. Prove that \( \sum_{k=\infty}^{\infty} f(k) = \sum_{k=\infty}^{\infty} \hat{f}(k) \).

   *Hint:* Consider the Fourier series of the periodic function \( F(x) = \sum_{k=-\infty}^{\infty} f(x+k) \).

   (b) Conclude that the Gaussian sum \( G(t) = \sum_{k=-\infty}^{\infty} e^{-\pi t k^2} \) satisfies \( G(t) = t^{-1/2} G(t^{-1}) \).

   In particular, \( G(t) \sim t^{-1/2} \) as \( t \to 0 \).

3. **The fundamental lemma of the Calculus of Variations**
   Prove: If \( F \in (C_c^\infty)'(\mathbb{R}) \) satisfies \( \frac{d}{dx} F = 0 \) in the sense of distributions, then \( F \) is (represented by) a constant function, i.e., there exists a constant \( c \) such that \( F(\phi) = c \int \phi \) for all test functions \( \phi \in C_c^\infty \).

   *Hint:* First consider the case where \( \int \phi = 0 \). (Write \( \phi = d\Phi/dx \) for some test function \( \Phi \).)

4. *(Folland, Exercise 8.16)*
   By the Riemann-Lebesgue lemma, the Fourier transform \( \mathcal{F} : f \mapsto \hat{f} \) defines a bounded linear transformation from \( L^1(\mathbb{R}^n) \) to the space \( C_0(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{C} \mid f \text{ continuous and } \lim_{|x| \to \infty} f(x) = 0 \right\} \) with the topology of uniform convergence. Convince yourself that \( C_0(\mathbb{R}^n) \) is complete.

   Let \( n = 1, t > 0 \), and consider the function \( f_t = \chi_{[-1,1]} * \chi_{[-t,t]} \).

   (a) Show that \( f_t \in C_0 \) and \( ||f_t||_\infty \leq 2 \). (Please make a sketch.)

   (b) But \( \lim_{t \to \infty} ||\hat{f}_t||_1 = \infty \), and likewise for \( \hat{f}_t \).

   (c) Argue with the open mapping theorem that \( \mathcal{F} : L^1 \to C_0 \) cannot be surjective.