MAT 1001 / 458 : Real Analysis II
Assignment 2, due January 23, 2013

1. (Operator norms for matrices)
   Let \( M \) be a \( n \times m \) matrix, and let \( L : \mathbb{R}^m \rightarrow \mathbb{R}^n \) be the linear transformation given by \( L(x) = Mx \). Compute the operator norm of \( L \), if both spaces are equipped with the \( \ell^p \)-norm, for \( p = 1, p = 2 \), and \( p = \infty \). (Use the same \( p \) for both spaces.)

2. (Folland 6.12)
   Prove that \( L^p(\mathbb{R}^n) \) is separable for \( 1 \leq p < \infty \) but not for \( p = \infty \).

3. (Folland 6.16)
   If \( 0 < p < 1 \), show that the formula \( \rho(f, g) = \int |f - g|^p \) defines a metric on \( L^p \) that makes \( L^p \) into a complete topological vector space. (You need to verify the triangle inequality, the continuity of translation and dilation, and completeness.)

4. (Heat kernel estimates)
   For \( t > 0 \) and \( x, y \in \mathbb{R}^n \), let \( K(t; x, y) = (4\pi t)^{-n/2}e^{-\frac{|x-y|^2}{4t}} \).
   (a) Given \( f \in L^1(\mathbb{R}^n) \), show that
   \[
   u(t, x) = \int_{\mathbb{R}^n} f(y) K(t; x, y) \, dy
   \]
solves the heat equation \( \partial_t u = \Delta u \) for \( t > 0 \). Here, the Laplacian operator \( \Delta \) is defined by \( \Delta u = \sum (\partial_{x_i})^2 u \).
   (b) Furthermore, \( \lim_{t \to 0^+} u(x, t) = f(x) \) for almost every \( x \in \mathbb{R}^n \).
   (c) Prove that \( u(t, \cdot) \in L^p(\mathbb{R}^n) \) for every \( p \in [1, \infty] \) and give a bound on its norm. (Hint: Interpolation.)
   (d) For which values of \( p \) is \( u \in L^p((0, T) \times \mathbb{R}^n) \)?