1. (Folland 3.13) Consider the unit interval $X = [0, 1]$ equipped with the Borel $\sigma$-algebra. Let $m =$ Lebesgue measure, and $\mu =$ counting measure. Prove that

(a) $m \ll \nu$, but $dm \neq f d\mu$ for any function $f$;

(b) $\mu$ has no Lebesgue decomposition with respect to $m$.

Why does that not contradict the Lebesgue-Radon-Nikodym theorem?

2. Let $\{f_n\}_{n \geq 1}$, $f, g$ be functions in $L^2[0, 2\pi]$, with $f_n \to f$ pointwise a.e. If $||f_n||_{L^2} \leq M$ for all $n$ and $g$ is bounded, then

$$\lim_{n \to \infty} \int_0^{2\pi} f_n(x) g(x) \, dx = \int_0^{2\pi} f(x) g(x) \, dx.$$ 

3. As in Problem 4 of Assignment 4, let $(x_n)_{n \geq 1}$ be the decimal expansion of $x \in (0, 1)$. (If the expansion is non-unique, take the one that terminates in 0.) You will show that

$$\lim_{n \to \infty} \left( \frac{1}{n} \# \{i = 1, \ldots, n : x_i = 7 \} \right) = 0.1$$

for almost every $x \in (0, 1)$.

(a) Let $y_n(x) = \chi_{(x_n = 7)} - 0.1$ and $S_n(x) = \sum_{k=1}^{n} y_n(x)$. Check that

$$\int_{(0,1)} y_m = 0, \quad \int_{(0,1)} y_m y_n = 0 \quad \text{for} \ m \neq n, \quad \text{and} \ \int_{(0,1)} y_n^2 \leq 1.$$ 

Use this to estimate $\int S_n^4$.

(b) Show that

$$\int_{(0,1)} \sum_{n=1}^{\infty} \left( \frac{S_n(x)}{n} \right)^4 < \infty,$$

and conclude that $S_n(x)/n \to 0$ for almost every $x$. 

1
4. *(Kolmogorov’s criterion)* Let \((\Omega, \mathcal{M}, \mu)\) be a probability space. A sequence of random variables \(X_i : \Omega \rightarrow \mathbb{R}\), for \(i = 1, 2, \ldots\) is called independent, if for every \(N > 0\) and every \(t_1, \ldots, t_N \in \mathbb{R}\),

\[
P(X_1 > t_1, \ldots, X_n > t_n) = \prod_{i=1}^{N} P(X_i > t_i) .
\]

If \((X_i)_{i \geq 1}\) is a sequence of independent random variables with \(E(X_i) = 0\) for all \(i\) and

\[
\sum_{i=1}^{\infty} E(X_i^2) < \infty ,
\]

prove that

\[
P \left( \sum_{i=1}^{\infty} X_i \text{ converges} \right) = 1 .
\]

5. *(Convolution with a smooth kernel)* Let \(\phi\) be a smooth complex-valued function \(\mathbb{R}^d\) with compact support (i.e., \(\phi\) vanishes outside some compact set \(K \subset \mathbb{R}^d\)). If \(f\) is integrable, prove that the convolution

\[
f * \phi(x) = \int f(x - y)\phi(y) \, dy
\]

is smooth. Moreover,

\[
\lim_{|x| \rightarrow \infty} f * \phi(x) = 0 .
\]

6. *(Lieb & Loss Problem 2.10)*

(a) Let \(f\) be a measurable real-valued function on the real line that is additive i.e.,

\[
f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.
\]

Prove that there exists an \(\alpha \in \mathbb{R}\) such that \(f(x) = \alpha x\), i.e., \(f\) is linear.

(b) Give an example of a (non-measurable) function that is additive but not linear.