MAT 1000 / 457 : Real Analysis I
Assignment 8, due November 20, 2013

1. \textit{(Uniform convergence of Fourier series)} Let \( f \) be a continuously differentiable \( 2\pi \)-periodic function, and let \((a_k)_{k \in \mathbb{Z}}\) be the sequence of its Fourier coefficients.

(a) Show that the series \( \sum_{k \in \mathbb{Z}} k^2|a_k|^2 \) converges.

(b) Use Schwarz’ inequality to verify that the sequence of partial sums \((S_n)\), given by

\[
S_n = \sum_{k=-n}^{n} a_k e^{-ikx}
\]

satisfies the Cauchy criterion with respect to the supremum norm, \(||g||_{\text{sup}} = \sup_{x} |g(x)|\). Hence the Fourier series converges uniformly to \( f \).

2. Find the values of

\[
\begin{align*}
\text{(a) } & \sum_{k=1}^{\infty} \frac{1}{k^2} \\
\text{(b) } & \sum_{k=1}^{\infty} \frac{1}{k^4} \\
\text{(c) } & \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}
\end{align*}
\]

\textit{Hint:} Compute the Fourier series of the function \( f(x) = x \) for \( x \in (-\pi, \pi) \).

3. \textit{(The Dirichlet kernel)} For \( n \in \mathbb{N} \), let \( P_n \) be the projection in \( L^2(0, 2\pi) \) defined by

\[
P_n f(x) = \sum_{k=-n}^{n} a_k e^{ikx}.
\]

Here, \((a_k)\) is the sequence of Fourier coefficients of \( f \). Show that

\[
P_n f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x - y) f(y) \, dy,
\]

where the integral kernel is given by

\[
D_n(x) = \frac{\sin((n + 1/2)x)}{\sin(x/2)}.
\]

4. Let \((c_k)_{k \in \mathbb{Z}}\) be a bi-infinite sequence of complex numbers that is square summable,

\[
\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty.
\]

Prove that \((c_k)\) is the sequence of Fourier coefficients of some \(2\pi\)-periodic function \( f \in L^2 \).
5. *(Fractional integrals, Folland 2.61)* If $f$ is continuous on $[0, \infty)$, for $\alpha > 0$ and $x \geq 0$ let

$$I_\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) \, dt.$$ 

(a) Prove that $I_{\alpha+\beta} f = I_\alpha(I_\beta f)$.

*Hint:* Use Problem 6 from Assignment 7 / Folland 2.60.

(b) If $n \in \mathbb{N}$, then $I^n f$ is an $n$-th order antiderivative of $f$.

6. *(Intermediate values for Lebesgue measure)* Let $A \subset B$ be compact sets in $\mathbb{R}^d$, and fix $t$ with $m(A) < t < m(B)$. Prove that there exists a compact set $K$ with $A \subset K \subset B$ and $m(K) = t$. 

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