1. (a) (Parallelogram identity) Let $\langle \cdot, \cdot \rangle$ be an inner product on a complex vector space $V$, and consider the norm defined by $\|f\| = \sqrt{\langle f, f \rangle}$. Prove that
\[ \|f + g\|^2 + \|f - g\|^2 = 2 \left( \|f\|^2 + \|g\|^2 \right). \]
(b) Show that the parallelogram identity fails in $L^1(\mathbb{R})$. Hence the $L^1$-norm does not come from an inner product.

2. (a) Let $(f_n)$ be a sequence of integrable functions that converges in $L^1$ to some limiting function $f$,
\[ \lim_{n \to \infty} \int |f_n - f| \, d\mu = 0. \]
Prove that there is a subsequence $(f_{n_j})$ that converges pointwise almost everywhere to $f$.
(b) (‘Typewriter’ sequences) Construct an example of a nonnegative sequence of integrable functions $(f_n)$ on the unit interval that converges to zero in $L^1$ such that the sequence of values $(f_n(x))$ converges for no $x \in (0, 1)$.

3. Let $(f_n)_{n \geq 1}$ be a sequence of functions on $[0, 1]$ that is bounded in $L^2$ (i.e., $\sup_n \|f_n\|_2 < \infty$). Assume that there exists a measurable function $f$ such that
\[ \lim_{n \to \infty} \int_0^1 |f_n - f| \, dm = 0 \quad (n \to \infty). \]
Show that $f \in L^2$. Does it follow that $f_n \to f$ in $L^2$?

4. (Lusin’s theorem) Let $E \subset \mathbb{R}^d$ be a set of finite measure, and let $f$ be a measurable real-valued (or complex-valued) function on $E$. Given $\varepsilon > 0$, prove that there exists a compact set $K \subset E$ with $m(E \setminus K) < \varepsilon$ such that the restriction $f|_K$ is continuous.

Hint: Approximate $f$ with a sequence of continuous functions and apply Egoroff’s theorem.
5. *(Folland 2.60: The Beta-integral)*

The Gamma-function is defined by

\[ \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \quad \text{for } x > 0. \]

Show that

\[ \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt \quad (x, y > 0). \]

*Hint:* Write \( \Gamma(x)\Gamma(y) \) as double integral and change variables in the inner integral to simplify the exponential.

6. For \( 1 \leq k \leq n \), compute the spherical integral

\[ \frac{1}{n\omega_n} \int_{S^{n-1}} (u_1^2 + \cdots + u_k^2)^{-1/2} \, d\sigma(u), \]

where \( \sigma \) is the standard rotationally invariant surface measure on the unit sphere \( S^{n-1} \).

*Hint:* Rewrite this as a Gaussian integral over \( \mathbb{R}^n \). Write your answer either in terms of the Gamma-function or in terms of the measures \( \omega_d \) of the \( d \)-dimensional unit balls.