1. \textit{(Folland 1.1)} A non-empty family of sets $\mathcal{R} \in \mathcal{P}(X)$ is called a \textbf{ring} if it is closed under finite unions and differences (i.e., if $E, F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$ and $E \setminus F \in \mathcal{R}$). A ring which is closed under countable unions is called a \textbf{σ-ring}.

(a) Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.
(b) Let $\mathcal{R}$ be a ring. Then $\mathcal{R}$ is an algebra, if and only if $X \in \mathcal{R}$.
(c) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.
(d) If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

2. Consider the collection of subsets of $\mathbb{N}$ that have a well-defined density,

$$C = \left\{ A \subset \mathbb{N} \left| \lim_{n \to \infty} \frac{1}{n} \#(A \cap \{1, \ldots, n\}) \right. \text{ exists} \right\}.$$  

Is $C$ a ring?

3. \textit{(Folland 1.3)} Let $\mathcal{M}$ be an infinite σ-algebra. Show that ...

(a) $\mathcal{M}$ contains an infinite sequence of disjoint non-empty sets;
(b) $\mathcal{M}$ is uncountable.

4. \textit{(Folland 1.4)} Let $\mathcal{A}$ be an algebra. Suppose that $\mathcal{A}$ is closed under countable increasing unions, i.e., $\bigcup_{j=1}^{\infty} E_j \in \mathcal{A}$ whenever $E_j \in \mathcal{A}$ and $E_j \subset E_{j+1}$ for each $j \in \mathbb{N}$.

Prove that $\mathcal{A}$ is a σ-algebra.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) \textit{(Inclusion-Exclusion, Folland 1.9)}

If $E, F \in \mathcal{M}$, then $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$.

(b) \textit{(Restricting a measure to a subset, Folland 1.10)}

Given a set $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Prove that $\mu_E$ is a measure.
6. (Folland 1.8) Let \((X, \mathcal{M}, \mu)\) be a measure space, and consider a sequence \((E_j)_{j \geq 1}\) in \(\mathcal{M}\). Define
\[
\liminf E_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j, \quad \limsup E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j.
\]

(a) Show that
\[
\liminf E_j = \{ x \mid x \in E_j \text{ for all but finitely many } j \}, \\
\limsup E_j = \{ x \mid x \in E_j \text{ for infinitely many } j \}.
\]
Conclude that \(\liminf E_j \subset \limsup E_j\).

(b) Give an example of a sequence \((E_j)\) where \(\liminf E_j \neq \limsup E_j\).

(c) Show that \(\mu(\liminf E_j) \leq \liminf \mu(E_j)\).
If \(\mu(\bigcup_{j=1}^{\infty} E_j) < \infty\), then also \(\mu(\limsup E_j) \geq \limsup \mu(E_j)\).