ASSIGNMENT №3

REAL ANALYSIS

MAT 1000

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[SS] := Stein and Shakarchi’s *Real Analysis*

**Problem 1 (exercise 21, page 43 in [SS])**

Let $\mathcal{C}$ be the standard middle-thirds Cantor set, and let $F : \mathcal{C} \rightarrow [0, 1]$ be the Cantor-Lebesgue function of exercise 2 in [SS]. Let $\mathcal{N}$ be the non-measurable set that is constructed in §1.3 of [SS], page 24.

Recall that the Cantor-Lebesgue function $F$ is surjective and $\mathcal{N} \subseteq [0, 1]$, so let

$$L := F^{-1}(\mathcal{N})$$

be the (non-empty) pre-image of $\mathcal{N}$ under $F$. Notice that $L \subseteq \mathcal{C}$, because $\mathcal{N} \subseteq [0, 1]$. Thus, the set $L$ is measurable, because it is a subset of the Cantor set $\mathcal{C}$, which is a set of measure zero. Finally, consider the restriction $F_L$ of $F$ to $L$:

$$F_L : L \rightarrow \mathcal{N}.$$ 

It is continuous as a restriction of a continuous function $F$, and it is maps a measurable set to a non-measurable set.

**Problem 2 (exercise 27, page 43 in [SS])**

Consider cubes $Q_t$ of side length $t \geq 0$ centred at the origin:

$$Q_t := \left[ -\frac{t}{2}, \frac{t}{2} \right]^d.$$

Then $m(Q_t) = t^d$ and observe that $Q_{t_1} \subseteq Q_{t_2}$ for $t_1 \leq t_2$. Now, for each $t \geq 0$, define a set

$$E(t) := (E_2 \cap Q_t) \cup E_1.$$ 

Notice that $E(t)$ is measurable as a finite intersection and union of measurable sets, as well as that $E(t_1) \subseteq E(t_2)$ for $t_1 \leq t_2$. More importantly, observe that $E(t)$ is compact (as a finite union of compact sets) with the property that

$$E_1 \leq E(t) \leq E_2.$$

In fact, $E(0) = E_1$, and there exists some $t_0$, such that $E(t_0) = E_2$, because $E_2$ is compact (hence bounded). Thus,

Now, define a real-valued function $f$ by

$$f(t) := m(E(t)).$$ 

Since the Lebesgue measure $m$ is a monotone function, $f$ is a monotone function, too: $f(t_1) \leq f(t_2)$ for $t_1 \leq t_2$. Moreover, $f(0) = a$ and $f(t_0) = b$, so it only remains to show that $f$ is a continuous function, because the result then follows immediately from the intermediate
value theorem (for any $c$ between $a$ and $b$, there exists some $\theta$ between 0 and $t_0$, such that $E(\theta)$ has measure $m(E(\theta)) = f(\theta) = c$).

To show that $f$ is continuous, let $\varepsilon > 0$ be given. For any $s > t$, we have

$$|f(s) - f(t)| = m(E(s)) - m(E(t)) = m(E(s) \setminus E(t)) \leq m(Q_s \setminus Q_t),$$

where the last inequality follows from the fact that $(E(s) \setminus E(t)) \subseteq (Q_s \setminus Q_t)$ and monotonicity of $m$. But then

$$m(Q_s \setminus Q_t) = m(Q_s) - m(Q_t) = s^d - t^d.$$

The function $g(s) = s^d$ is absolutely continuous, so there exists some $\delta > 0$, such that, if $|s - t| < \delta$, we have $|g(s) - g(t)| < \varepsilon$, i.e.

$$|f(s) - f(t)| \leq |s^d - t^d| = |g(s) - g(t)| < \varepsilon,$$

so $f$ is indeed continuous.

**Problem 3 (exercise 37, page 45 in [SS])**

By translational invariance and countable subadditivity, it is enough to consider the case

$$f : [0, 1] \to \mathbb{R}.$$

Let $\varepsilon > 0$ be given. Now, any continuous function on a compact domain is uniformly continuous, so there exists some $\delta > 0$, such that, for all $x, y$ satisfying $|x - y| < \delta$, we must have $|f(x) - f(y)| < \varepsilon$. Let $\delta_0 := \min \{\delta, \varepsilon\}$, and cover $\Gamma$ by rectangles of width $\delta_0$ and height $\varepsilon$. Namely, let $N \in \mathbb{N}$ be such that $N\delta_0 \leq 1 < (N + 1)\delta_0$, and consider the following rectangles:

$$R_k := [k\delta_0, (k + 1)\delta_0] \times \left[ f(k\delta_0) - \frac{\varepsilon}{2}, f(k\delta_0) + \frac{\varepsilon}{2} \right], \quad \text{for } k = 0, \ldots, N$$

By uniform continuity of $f$, it is clear that

$$\Gamma \subseteq \bigcup_{k=1}^{N} R_k,$$

and the union is measurable as a finite union of measurable sets. Therefore, by monotonicity,

$$m_*(\Gamma) \leq m\left( \bigcup_{k=1}^{N} R_k \right) \leq \sum_{k=1}^{N} m(R_k) \leq \sum_{k=1}^{N} \varepsilon\delta_0 = N \varepsilon \delta_0 \leq \varepsilon,$$

where in the last inequality we have used the fact that $N\delta_0 \leq 1$ by the choice of $N$. Thus, as $\varepsilon \to 0$, $m_*(\Gamma) \to 0$. In particular, then, $\Gamma$ is measurable and $m(\Gamma) = 0$. 

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