Question 1  Find the principal part of the Laurent Series and identify the residue for

\[ f(z) = \frac{\sin z^2}{z^3(1 + z)} \]

at \( z = 0 \).

Solution  We know the power series for \( \sin z \) and \( 1/(1 - z) \) already, so we have for free that

\[ \sin z^2 = z^2 - \frac{z^6}{3!} + \mathcal{O}(z^{10}) \]
\[ \frac{1}{1 + z} = 1 - z + \mathcal{O}(z^2) \]

Thus by simple multiplication we see

\[ f(z) = \frac{\sin z^2}{z^3(1 + z)} \]
\[ = \frac{1}{z^3} \left( z^2 - \frac{z^6}{3!} + \mathcal{O}(z^{10}) \right) (1 - z + \mathcal{O}(z^2)) \]
\[ = \frac{-1 + z + \mathcal{O}(z^2)}{z^3} \]

Thus we can read off the principal part, and we see the coefficient on the \( 1/z \) term is 1, which is also the residue of \( f(z) \) at \( z_0 = 0 \).

Question 2  Derive the Laurent series representation

\[ \frac{e^z}{(z + 1)^2} = \frac{1}{e} \left( \sum_{n=0}^{\infty} \frac{(z + 1)^n}{(n + 2)!} + \frac{1}{z + 1} + \frac{1}{(z + 1)^2} \right), \quad z \neq 1 \]

Solution  We first compute the power series of \( e^z \) entered around \( z = -1 \) using the trick from last week. We have

\[ e^z = \frac{1}{e} e^{z+1} = \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z + 1)^n}{n!} \]

Thus

\[ \frac{e^z}{(z + 1)^2} = \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z + 1)^{n-2}}{n!} = \frac{1}{e} \left( \sum_{n=0}^{\infty} \frac{(z + 1)^n}{(n + 2)!} + \frac{1}{z + 1} + \frac{1}{(z + 1)^2} \right), \quad z \neq 1 \]
Question 3  Evaluate
\[ \int_{|z|=1} \cos \left( \frac{1}{z^2} \right) e^{1/z} \, dz \]

Solution  We know the power series representation about \( z_0 = 0 \) for both functions, thus we see
\[
\cos \left( \frac{1}{z^2} \right) e^{1/z} = \left( 1 - \frac{1}{2!} \frac{1}{z^4} + \mathcal{O} \left( \frac{1}{z^8} \right) \right) \left( 1 + \frac{1}{z} + \frac{1}{2z^2} + \mathcal{O} \left( \frac{1}{z^3} \right) \right)
\]
\[= 1 + \frac{1}{z} + \frac{1}{2z^2} + \mathcal{O} \left( \frac{1}{z^3} \right) \]

As we already know, we have that
\[ \int_{|z|=1} \frac{dz}{z^n} = \begin{cases} 
2\pi i & n = 1 \\
0 & n \neq 1 
\end{cases} \]

Thus we see
\[ \int_{|z|=1} \cos \left( \frac{1}{z^2} \right) e^{1/z} \, dz = 2\pi i \]

Alternate Solution  Take a change of variables to the integrand, \( w = 1/z, \) we see
\[ dw = -\frac{dz}{z^2}, \]

therefore
\[ \int_{|z|=1} \cos \left( \frac{1}{z^2} \right) e^{1/z} \, dz = \int_{|w|=1} \frac{\cos (w^2) e^w}{w^2} \, dw \]

note the double negative via the change of orientation. By expanding out both terms into power series, we see
\[ \int_{|w|=1} \frac{\cos (w^2) e^w}{w^2} \, dw = \int_{|w|=1} \frac{1}{w^2 \left( 1 - \frac{w^2}{2} + \mathcal{O}(w^4) \right) \left( 1 + w + \frac{w^2}{2} + \mathcal{O}(w^3) \right)} \, dw = \int_{|w|=1} \left( \frac{1}{w^2} + \frac{1}{w} + \mathcal{O}(w) \right) \, dw \]

Thus we see
\[ \int_{|w|=1} \left( \frac{1}{w^2} + \frac{1}{w} + \mathcal{O}(w) \right) \, dw = 2\pi i \]

Question 4  Evaluate the integral using Cauchy’s Residue theorem.
\[ \int_{C} \frac{\exp(-z)}{\sin(z^2 + z)} \, dz \]

where \( C = \{ z : |z| = 3/2 \}. \)

Solution  We know \( e^{-z} \) is an entire function, thus we have to find the zeros of \( \sin(z^2 + z) \) (i.e. the poles in question) to apply the residue theorem. We see
\[ \sin(z(z+1)) = 0 \implies z^2 + z = n\pi, \quad n \in \mathbb{Z} \implies z = \frac{-1 \pm \sqrt{1 + 4n\pi}}{2} \]

Since \( C \) is just the circle of radius 1 centred at 0, we see 2 poles are inside, specifically at \( z = -1 \) and \( z = 0. \) We calculate the residues:
\[ \text{Res}(f, 0) = \lim_{z \to 0} z \frac{\exp(-z)}{\sin(z^2 + z)} = \lim_{z \to 0} \frac{z}{\sin(z^2 + z)} = \lim_{z \to 0} \frac{1}{(2z + 1) \cos(z^2 + z)} = 1 \]
\[
\text{Res}(f, -1) = \lim_{z \to -1} (z + 1) \frac{\exp(-z)}{\sin(z^2 + z)} = \lim_{z \to -1} \frac{e(z + 1)}{\sin(z^2 + z)} = \lim_{z \to -1} \frac{e}{(2z + 1) \cos(z^2 + z)} = -e
\]

The residue theorem now tells us
\[
\int_{C} \frac{\exp(-z)}{\sin(z^2 + z)} \, dz = 2\pi i (1 - e)
\]

**Question 5** Calculated the integral
\[
\int_{C} \frac{\cos(\pi z)}{\sin(\pi z)(1 + z^4)} \, dz
\]
where \(C\) is the counterclockwise oriented rectangle with vertices at \(z_{1,2} = \pm \frac{3+i}{2}, z_{3,4} = \pm \frac{3-i}{2}\)

**Solution** We see the poles of the function in question are given by
\[
\sin(\pi z) = 0 \implies z = k \quad k \in \mathbb{Z} \quad \& \quad 1 + z^4 = 0 \implies z = \exp \left( i \pi \frac{k}{4} \right), \quad k = 0, 1, 2, 3
\]
If we check which poles lie inside \(C\), we see the only poles we have to consider are \(z = 0, \pm 1\). We can now compute the residues and use the residue theorem. Note that all the poles are simple, thus we have
\[
\text{Res}(f, 0) = \lim_{z \to 0} z \frac{\cos(\pi z)}{\sin(\pi z)(1 + z^4)} = \lim_{z \to 0} \frac{z}{\sin(\pi z)} = \frac{1}{\pi}
\]
\[
\text{Res}(f, \pm 1) = \lim_{z \to \pm 1} (z \pm 1) \frac{\cos(\pi z)}{\sin(\pi z)(1 + z^4)} = \lim_{z \to \pm 1} \frac{z \pm 1}{2 \sin(\pi z)} = \frac{1}{2\pi}
\]
Thus we have
\[
\int_{C} \frac{\cos(\pi z)}{\sin(\pi z)(1 + z^4)} \, dz = 2\pi i \left( \text{Res}(f, 0) + \text{Res}(f, 1) + \text{Res}(f, -1) \right) = 4i
\]