5.1 - # 15  Find the Laplace Transform of

\[ f(t) = \begin{cases} 
0, & 0 \leq t \leq 1 \\
1, & 1 < t \leq 2 \\
0, & 2 < t 
\end{cases} \]

Solution  By direct computation we have

\[ \mathcal{L}\{f(t)\} = \int_1^2 e^{-st} dt = \frac{-1}{s} e^{-st}\bigg|_1^2 = \frac{e^{-s} - e^{-2s}}{s} \]

5.1 - # 16  Find the Laplace Transform of

\[ f(t) = \begin{cases} 
0, & 0 \leq t \leq 1 \\
e^{-t}, & 1 < t 
\end{cases} \]

Solution  Again, by direct computation we have

\[ \mathcal{L}\{f(t)\} = \int_1^\infty e^{-st} e^{-t} dt = \int_1^\infty e^{-(s+1)t} dt = \frac{-1}{s+1} e^{-(s+1)t}\bigg|_1^\infty = \frac{e^{-(s+1)}}{s+1} \]

5.1- # 37  Consider the Laplace transform of \( t^p \), where \( p > -1 \).

(a)  Referring to Problem 36, show that

\[ \mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p dt = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx = \frac{\Gamma(p+1)}{s^{p+1}}. \quad s > 0 \]

Proceed with the change of variables \( x = st \), treating \( s \) as a positive constant (otherwise we’ll have orientation issues). We obtain

\[ \mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p dt = \int_0^\infty e^{-x} \left( \frac{x}{s} \right)^p \frac{1}{s} dx = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p dx \]

Following Problem 36 gives us the integral formulation for the Gamma function, hence

\[ \mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0 \]
(b) Let \( p \) be a positive integer \( n \) in a). Show that

\[
\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0
\]

If \( p \) is a positive integer \( n \), by Problem 36 we know that

\[
\Gamma(p + 1) = n!
\]

Using our above formula, now shows that

\[
\mathcal{L}\{t^n\} = \frac{\Gamma(n + 1)}{s^{n+1}} = \frac{n!}{s^{n+1}}, \quad s > 0
\]

(c) Show that

\[
\mathcal{L}\{t^{-1/2}\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{2}}, \quad s > 0
\]

Again, by problem 36, we know that \( \Gamma(1/2) = \sqrt{\pi} \), hence if \( p = -1/2 \) we have

\[
\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(1/2)}{s^{3/2}} = \sqrt{\frac{\pi}{s}}, \quad s > 0
\]

(d) Show that

\[
\mathcal{L}\{t^{1/2}\} = \frac{\sqrt{\pi}}{2s^{3/2}}
\]

Again, by problem 36, we have that \( p!\Gamma(p) = \Gamma(p + 1) \) for \( p > 0 \). So we have

\[
\mathcal{L}\{t^{1/2}\} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\Gamma(1/2)}{2s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}, \quad s > 0
\]

5.2- #11 Let \( F(s) = \mathcal{L}\{f(t)\} \), where \( f(t) \) is piecewise continuous and of exponential order on \([0, \infty)\). Show that

\[
\mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s)
\]

**Solution** Recall that we have

\[
\mathcal{L}\{f'(t)\} = s \mathcal{L}\{f(t)\} - f(0)
\]

via integration by parts since \( f(t) \) is piecewise continuous and of exponential order. Define

\[
g(t) = \int_0^t f(\tau) d\tau \implies g'(t) = f(t)
\]

Hence, by plugging this into the above formula we have

\[
\mathcal{L}\{f(t)\} = s \mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\} \implies \mathcal{L}\left\{ \int_0^t f(\tau) d\tau \right\} = \frac{F(s)}{s}
\]

5.2- #28 The Laplace transforms of certain functions can be found conveniently form their Taylor series expansions. Using the Taylor series for \( \sin t \)

\[
\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}
\]

and assuming that the Laplace transform of this series can be computed term by term, verify that

\[
\mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \quad s > 1
\]
Solution  By linearity, we have that
\[\mathcal{L}\{\sin t\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \mathcal{L}\{t^{2n+1}\}\]

Luckily, we previous did a question that gave us a nice formula for this! Recalling
\[\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{t^{2n+1}\} = \frac{(2n+1)!}{s^{2n+2}}\]

Plugging this into the above form, we have
\[\mathcal{L}\{\sin t\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n+2}} = \frac{1}{s^2} \sum_{n=0}^{\infty} \left(\frac{1}{s^2}\right)^n\]

Recall the geometric series formula
\[\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n\]

clearly, if \(x = -1/s^2\), then
\[\frac{1}{1+s^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{s^{2n+2}} = \mathcal{L}\{\sin t\}\]

as desired.

5.2 - # 29  For each of the following initial value problems, use Theorem 5.2.4 to find the differential equation satisfied by \(F(s) = \mathcal{L}\{f(t)\}\), where \(y = f(t)\) is the solution of the given initial value problem.
\[y'' - ty = 0; \quad y(0) = 1, \quad y'(0) = 0\]

Solution  Theorem 5.2.4 states that for \(f\) piecewise continuous of exponential order \(a\), then
\[\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad s > a\]

where \(F^{(n)} = d^n/dx^n \mathcal{L}\{f(t)\}\). Clearly we have
\[\mathcal{L}\{y''(t)\} = s^2 F(s) - sf(0) - f'(0) = s^2 F(s) - s\]

and
\[\mathcal{L}\{ty(t)\} = -F'(s)\]

Hence
\[\mathcal{L}\{y'' - ty\} = \mathcal{L}\{y''\} - \mathcal{L}\{ty\} = s^2 F(s) + F'(s) = s\]

is the ODE we’re looking for.

5.3 - # 17  Use the linearity of \(\mathcal{L}^{-1}\) with partial fraction expansion and Table 5.3.1 to find the inverse Laplace transform of the given function:
\[F(s) = \frac{-1 - 2s}{s^2 + 4s + 5}\]
Solution  

Well, by glancing at the table, we see it’s a pretty good idea to complete the square in the dominator, so we do.

\[ s^2 + 4s + 5 = (s + 2)^2 + 1 \]

Hence we may rewrite \( F(s) \) as

\[ F(s) = \frac{1 - 2s}{s^2 + 4s + 5} = 5 \left( \frac{1}{(s + 2)^2 + 1} \right) - 2 \left( \frac{s + 2}{(s + 2)^2 + 1} \right) \]

Noting that

\[ \mathcal{L}^{-1}\left\{ \frac{1}{(s + 2)^2 + 1} \right\} = e^{-2t} \sin(t) \quad \& \quad \mathcal{L}^{-1}\left\{ \frac{s + 2}{(s + 2)^2 + 1} \right\} = e^{-2t} \cos(t) \]

by the table, we have

\[ \mathcal{L}^{-1}\{F(s)\} = e^{-2t}(5 \sin t - 2 \cos t) \]

5.3 - # 24  
Use the linearity of \( \mathcal{L}^{-1} \) with partial fraction expansion and Table 5.3.1 to find the inverse Laplace transform of the given function:

\[ F(s) = \frac{s^2 + 3}{(s^2 + 2s + 2)^2} \]

Solution  

Rewrite the fraction as follows

\[ F(s) = \frac{s^2 + 3}{(s^2 + 2s + 2)^2} = \frac{(s + 1)^2 + 1 - 2s + 1}{((s + 1)^2 + 1)^2} = \frac{1}{(s + 1)^2 + 1} - \frac{2s - 1}{((s + 1)^2 + 1)^2} \]

Now the table manages the first term, and we see that the other term looks like some sort of derivative. So we try

\[ -\frac{d}{ds}\left( A \frac{1}{(s + 1)^2 + 1} + B \frac{s + 1}{(s + 1)^2 + 1} \right) = \frac{2A(s + 1) + Bs(s + 2)}{((s + 1)^2 + 1)^2} \]

We have to complete the square on the top for \( B \), so

\[ B \frac{s(s + 2)}{((s + 1)^2 + 1)^2} = B \frac{(s + 1)^2 + 1 - 2s}{((s + 1)^2 + 1)^2} = B \frac{1}{(s + 1)^2 + 1} - B \frac{2s}{(s + 1)^2 + 1} \]

Hence, comparing with our original expansion we see \( B = 3/2 \) and \( A = 1 \)

\[ F(s) = \left( \frac{5}{2} + \frac{d}{ds}\right) \left( \frac{1}{(s + 1)^2 + 1} \right) + \frac{3}{2} \frac{d}{ds} \left( \frac{s + 1}{(s + 1)^2 + 1} \right) \]

Comparing with the table gives

\[ \Rightarrow \mathcal{L}^{-1}\{F(s)\} = \left( \frac{5}{2} - t \right) e^{-t} \sin t - \frac{3}{2} te^{-t} \cos t \]