n-th Order Linear Differential Equations with Constant Coefficients

Always try $e^{\lambda x}$ as a solution to

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y^{(0)} = 0$$

since

$$\frac{d^n}{dx^n}e^{\lambda x} = \lambda^n e^{\lambda x} \implies e^{\lambda x} (a_n \lambda^n + a_{n-1} \lambda^{n-1} \ldots + a_0) = 0$$

Thus $e^{\lambda x}$ is a solution if (since the exponential is never zero)

$$P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} \ldots + a_0 = 0$$

i.e. $\lambda$ is a root of $P(\lambda)$, which is called the characteristic polynomial. By the fundamental theorem of algebra, we know that we’ll always find $n$ roots over the complex numbers. Thus we’ve found $n$ solutions to the ODE

**Why Is It Called The Characteristic Equation?**

Recall from last time, we saw

$$ay'' + by' + cy = 0 \iff \dot{x} = \begin{pmatrix} 0 & a \\ -c & -b \end{pmatrix} x \quad \text{where} \quad x = \begin{pmatrix} y \\ y' \end{pmatrix}$$

Notice that the characteristic equation, the one that determines the eigenvalues, is the same as the previous polynomial,

$$P(\lambda) = \det(A - 1\lambda) = \lambda^2 + \frac{b}{a} \lambda + \frac{c}{a} = 0 \implies a\lambda^2 + b\lambda + c = 0$$

**Repeated Roots**

On can check that you have repeated roots from your characteristic equation. To form a full basis for your solution space, i.e. the fundamental solutions, you can just stick $t$ in front of the exponential for every solution you’re missing. i.e. if $P(\lambda) = (\lambda - a)^3$, we’d have

$$y_1 = e^{at} \quad \& \quad y_2 = te^{at} \quad \& \quad y_3 = t^2 e^{at}$$

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Solve

$$y'' - 2ay' + a^2 y = 0$$
Solution  The characteristic polynomial for the ODE is
\[ P(\lambda) = \lambda^2 - 2a\lambda + a^2 = (\lambda - a)^2 \]
i.e. we have a repeated root of \( \lambda = a \). Let’s try \( y(t) = te^{at} \) as a solution,
\[
2a \left( \frac{e^{at} + a^2t e^{at}}{y''} \right) - 2a \left( \frac{e^{at} + ate^{at}}{y'} \right) + a^2 \left( \frac{te^{at}}{y} \right) = 0
\]
Thus the general solution is given by
\[ y(t) = c_1 e^{at} + c_2 te^{at} \]

Example  Solve the IVP
\[ 3y''' + 5y'' + y' - y = 0, \quad y(0) = 0, y'(0) = 1, y''(0) = -1 \]

Solution  The characteristic equation for the ODE is
\[ P(\lambda) = 3\lambda^3 + 5\lambda^2 + \lambda - 1 = (\lambda + 1)^2(3\lambda - 1) = 0 \implies \lambda = \frac{1}{3} \quad \text{&} \quad \lambda = -1 \text{ (Repeated)} \]
Thus general solution is given by
\[ y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 e^{t/3} \]
The initial data implies
\[
\begin{cases}
  c_1 + c_3 = 0 \\
  -c_1 + c_2 + c_3/3 = 1 \\
  c_1 + 2c_2 + c_3/9 = -1
\end{cases}
\implies c_1 = -\frac{9}{16}, c_2 = \frac{1}{4}, c_3 = \frac{9}{16}
\]
Hence the solution to the IVP is
\[ y(t) = \frac{9}{16} e^{t/3} + \left( \frac{t}{4} - \frac{9}{16} \right) e^{-t} \]

Complex Eigenvalues  In the case of complex eigenvalues, it seems we have a complex valued solution...though using Euler’s Identity
\[ e^{i\theta} = \cos \theta + i \sin \theta \]
we may rewrite the solution in terms of real valued functions. Let \( \lambda_+ = a + bi \) and \( \lambda_- = a - bi \), then
\[
y(t) = Ce^{\lambda_+ t} + \bar{C}e^{\lambda_- t} = e^{at}(Ce^{ibt} + \bar{C}e^{-ibt})
\]
\[
= e^{at}((C + \bar{C}) \cos(bt) + i(C - \bar{C}) \sin(bt))
\]
\[
= e^{at}(c_1 \cos(bt) + c_2 \sin(bt))
\]
Euler Equations  Consider
\[ ax^2y'' + bxy' + cy = 0 \quad x > 0 \]
At first glance it seems like a foreign equation, but let’s apply the change of variables \( x \to z = \ln(x) \) to the ODE. We see via chain rule that
\[
\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}
\]
Thus, the ODE in \( z \) becomes
\[
a \frac{d^2y}{dz^2} + (b - a) \frac{dy}{dz} + cy = 0
\]
i.e. an ODE with constant coefficients. We know the solutions take exponential form...or in terms of the variable \( x \) we have
\[
e^{xz} = e^{x \ln(x)} = e^{\ln(x^x)} = x^x
\]
Thus we see trying \( y(x) = x^x \) will amount to the same ole story.

HW - #1  Consider the equation
\[
y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_1y' + p_0y = 0
\]
with \( p_j(x) \) for each \( j \in [0, n-1] \) continuous on \([a, b]\). Suppose that \( y \) is a solution with infinitely many zeros in the interval \([a_1, b_1]\) such that \( a < a_1 < b_1 < b \). Prove that \( y \equiv 0 \) on \((a, b)\).

Proof  By the Bolzano Weierstrass Theorem, we know the infinite sequence of zeros \( \{x_m\}_{m=1}^\infty \) has a converging subsequence to \( x_0 \in [a_1, b_1] \). By definition of a solution, we require \( y \) and it’s derivatives to be continuous on \([a, b]\). This leads us to consider the neighbourhood \( B_\delta(x_0) = \{ x : x \in (x_0 - \delta, x_0 + \delta) \} \) for any \( \delta > 0 \), and check if \( y \neq 0 \) there. This will only happen if \( y'(x_0) \neq 0 \), but \( B_\delta(x_0) \) has infinitely many zero’s of \( y \), so this means that \( y' = 0 \) on \( B_\delta(x_0) \). Repeating this argument with \( y' \), and moving up to higher derivatives, we conclude
\[
y(x_0) = y'(x_0) = \ldots = y^{n-1}(x_0) = 0
\]
Next we’ll show that the trivial solution is the only solution to a null data problem. Define
\[
\xi = \sum_{k=0}^{n-1} (y^{(k)})^2 \geq 0
\]
Then the derivative will give us
\[
\xi' = 2 \sum_{k=0}^{n-1} y^{(k)}y^{(k+1)}
\]
Plug the ODE into the above
\[
\xi' = yy' + \ldots + y^{(n-2)}y^{(n-1)} + y^{(n-1)}(-p_{n-1}y^{(n-1)} - \ldots p_0y)
\]
Using the Inequality \( 2ab \leq a^2 + b^2 \) and the fact that \( p_j \) for each \( j \) is continuous (we may bound each above), we see
\[
\xi' \leq K\xi
\]
for some constant $K$. Gronwall’s Inequality now tells us

$$
\xi(x) \leq \xi(x_0)e^{K(x-x_0)} = 0, \quad x \geq x_0
$$

A similar bound is found for $\xi' \geq -K\xi$ (use $2ab \geq -a^2 - b^2$) to conclude that $\xi = 0$ for $x < x_0$, together these imply that $y \equiv 0$ on $[a, b]$.

**Quiz** Suppose that vector functions $y_1(x), \ldots, y_n(x)$ taking values in $\mathbb{R}^n$ are linearly independent on the interval $[a, b]$, and all their coordinates are differentiable on $[a, b]$. Show that there exists a matrix function $A(x) = (a_{ij}(x))_{1 \leq i, j \leq n}$ such that $y_1(x), \ldots, y_n(x)$ are solution of the system $y' = Ay$ on $[a, b]$.

**Solution** Define the matrix $X$ to have columns $y_j$:

$$
X(x) = (y^1, \ldots, y^n)
$$

Then we’d like this matrix to solve

$$
X' = AX
$$

So we now have an equation for the $A(x)$ we’d like to construct, we simply set

$$
X'(x)X^{-1}(x) = A(x)
$$

Note the inverse is well defined since all vectors are linearly independent. This is such an $A$. 

\[ \Box \]