pg.56-# 19 Solve
\[\begin{align*}
(1 - x)dy &= x(y + 1)dx \\
y(0) &= 0
\end{align*}\]

**Solution** The equation is separable, thus
\[
\frac{dy}{y + 1} = \frac{x \, dx}{1 - x} \implies \int \frac{dy}{y + 1} = \int \frac{x \, dx}{1 - x} \implies \ln |y + 1| = \ln \left| \frac{1}{1 - x} \right| + x + C \implies y(x) = \frac{\hat{C}e^x}{1 - x} - 1
\]
is the general solution. The initial condition fixes the constant.

\[y(0) = 0 \implies 0 = \hat{C} - 1 \implies \hat{C} = 1\]

Thus the solution to the IVP is
\[y(x) = \frac{e^x}{1 - x} - 1\]

pg.69 - #6 Solve
\[(x + y)dx + (2x + 2y - 1)dy = 0\]

**Solution** We see the lines in the equation are parallel since
\[u = x + y \implies 2u - 1 = 2x + 2y - 1\]

Thus we make the change of variables \(u\) as above, we see \(du = dx + dy\) and
\[(1 - u)dx + (2u - 1)du = 0\]
is a separable equation. Thus the implicit solution is given by
\[
\int dx = \int \frac{1 - 2u}{1 - u} \, du = \int \frac{1}{1 - u} + \int \left(2 - \frac{2}{1 - u}\right) \, du \implies x + C = \ln |1 - u| + 2u - 2 \ln |1 - u| \implies x + 2y - \ln |x + y - 1| = C
\]
pg.69 - #10  Solve

\[(3x - 2y + 4)dx - (2x + 7y - 1)dy = 0\]

**Solution**  We know it is possible to convert this to a homogeneous equation, we just need to find the change of variables. We’ll use the differential method, i.e. we want \(u\) and \(v\) such that

\[3x - 2y + 4 = u \quad \& \quad -2x - 7y + 1 = v\]

This implies that

\[
\begin{align*}
\frac{du}{dv} &= \frac{3}{-2} \\
\frac{dv}{du} &= \frac{-2}{-7} \\
\frac{3}{-2} &= \frac{-2}{-7}
\end{align*}
\]

By inverting the matrix we see that

\[
\frac{1}{25} \begin{pmatrix} 7 & -2 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} dx \\ dy \end{pmatrix}
\]

Thus the ODE becomes the following in coordinates \(u\) and \(v\),

\[
u dx + v dy = u \left( \frac{7}{25} du - \frac{2}{25} \right) + v \left( -\frac{2}{25} du - \frac{3}{25} dv \right) = \frac{1}{25} [(7u - 2v)du + (-2u - 3v)dv]
\]

Now the equation is homogeneous, which we know is possible to solve using \(u = tv\), \(du = tdv + vdt\). Substitute the change of variables again. (note we’ve dropped the \(1/25\) since the RHS is 0)

\[(7tv - 2v)(tdv + vdt) + (-2tv - 3v)dv = (7t^2 - 4t - 3)vdt + (7t - 2)v^2 dt = 0\]

This equation is separable, and we see the solution is \((\omega = 7t^2 - 4t - 3, d\omega = 2(7t - 2)dt)\)

\[
\int \frac{dv}{v} = \int \frac{(2 - 7t)dt}{7t^2 - 4t - 3} = -\frac{1}{2} \int \frac{d\omega}{\omega} \implies \ln|\omega^2| = C
\]

Now we just back substitute everything to revert to the original coordinates.

\[\omega v^2 = \tilde{C} \implies 7(vt)^2 - 4tv^2 - 3v^2 = \tilde{C} \implies 7u^2 - 4uv - 3v^2 = \tilde{C}\]

Hence the implicit general solution is

\[7y^2 + (2 - 4x)y + 3x^2 + 8x = \text{Const}\]

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pg.79 - #16  Solve

\[
\begin{align*}
\sin x \cos y + \cos x \sin y dy &= 0 \\
y(\pi/4) &= \pi/4
\end{align*}
\]
Solution  By the symmetry of the equation, we check if it is exact. i.e
\[ M_y = N_x \] where \[ \sin x \cos y \frac{dx}{M} + \cos x \sin y \frac{dy}{N} = 0 \]

Clearly
\[ M_y = - \sin x \sin y = N_x \]

Thus the equation is exact, the solution is therefore given by a level set of linearity independent factors of the integrated functions. I write this as
\[ F(x, y) = \int Mdx \oplus \int Ndy = \int \sin x \cos y dx \oplus \int \cos x \sin y dy = 0 \]

Thus the general solution is
\[ \cos x \cos y = C \]

The initial data implies that
\[ C = \cos(\pi/4) \cos(\pi/4) = \frac{1}{2} \Rightarrow 1 = 2 \cos x \cos y \]

is the implicit solution to the IVP.

pg.79 - #12  Solve
\[ x \sqrt{x^2 + y^2} dx - \frac{x^2 y}{y - \sqrt{x^2 + y^2}} dy = 0 \]

Solution  Notice we may rewrite the 2nd component since
\[ - \frac{x^2 y}{y - \sqrt{x^2 + y^2}} = - \frac{x^2 y}{y - \sqrt{x^2 + y^2}} + \frac{y}{y - \sqrt{x^2 + y^2}} = - \frac{x^2 y^2 + \sqrt{x^2 + y^2}}{y^2 - \sqrt{x^2 + y^2}} = y^2 + y \sqrt{x^2 + y^2} \]

It’s easy to check that the equation is exact since
\[ M_y = \frac{xy}{\sqrt{x^2 + y^2}} = N_x \]

Thus solution is given by
\[ F(x, y) = \int Mdx \oplus \int Ndy = \int x \sqrt{x^2 + y^2} dx \oplus \int (y^2 + y \sqrt{x^2 + y^2}) dy = \frac{1}{3}(x^2 + y^2)^{3/2} + \frac{y^3}{3} + \frac{1}{3}(x^2 + y^2)^{3/2} \]

Thus the implicit solution is given by
\[ (x^2 + y^2)^{3/2} + y^3 = \text{const} \]
Quiz  Solve

\[ ydy + xdx = 3xy^2dx, \quad y(2) = 1 \]

Solution  Rewrite the ODE to

\[ ydy = x(3y^2 - 1)dx \]

Clearly this equation is separable, thus the implicit solution is given by

\[
\int \frac{y}{3y^2 - 1}dy = \int xdx \implies \frac{1}{6} \ln|3y^2 - 1| = \frac{x^2}{2} + C \implies 3y^2 - 1 = \tilde{C}e^{3x^2} \implies y = \pm \sqrt{\tilde{C}e^{3x^2} + \frac{1}{3}}
\]

The initial data implies that we need the + sign, and

\[ \tilde{C} = \frac{2e^{-12}}{3} \implies y = \frac{1}{\sqrt{3}} \sqrt{2e^{3(x^2 - 4)}} + 1 \]