The following problems build on each other. If you can’t figure out a problem, feel free to assume it and move on to the next one.

1. Let \( p \) be an odd prime, and let \( K_p = \mathbb{Q}[x]/(f_p(x)) \) where
\[
f_p(x) = \sum_{i=0}^{p-1} x^i.
\]
Prove that \( f_p(x) \) is irreducible using Eisenstein’s criterion for \( f_p(x + 1) \). Hence \( K_p \) is a field. It is called the \( p \)'th cyclotomic field.

2. Prove that \( K_p \) is Galois over \( \mathbb{Q} \) with Galois group naturally isomorphic to \((\mathbb{Z}/p\mathbb{Z})^\times \) given by \( a \to \sigma_a \) where \( \sigma_a(x) = x^a \).

3. Prove that \( (1 - x) \) has norm \( p \), and that \( (1 - x)^p \) is divisible by \( p \) in the ring of integers \( \mathcal{O}_{K_p} \). Conclude that \( p \) is totally ramified in \( K_p \).

4. By computing the Discriminant of \( \mathbb{Z}[x] \), prove that \( \mathbb{Z}[x] \) has index a power of \( p \) in \( \mathcal{O}_{K_p} \).

5. Using the basis \( (1 - x)^i, i = 0, 1, 2 \ldots, p - 2 \), or otherwise, prove that \( \mathbb{Z}[x] = \mathcal{O}_{K_p} \).

6. Let
\[
p_s = \begin{cases} 
p & \text{if } p \equiv 1 \mod 4 \\
-p & \text{otherwise}.
\end{cases}
\]
Prove that \( \mathbb{Q}(\sqrt{p_s}) \) is the unique quadratic subfield of \( K_p \). \textbf{Hint: Use Galois theory, and in particular the automorphism in the Galois Group corresponding to complex conjugation}.

7. For a prime number \( q \neq p \), prove that \( q \) is unramified, prove that the number of primes above \( q \) in \( K_p \), and the size of their decomposition groups, depend only on \( q \mod p \).

8. Using the above, prove that \( q \) splits in \( \mathbb{Q}(\sqrt{p_s}) \) iff \( q \) is a quadratic residue modulo \( p \).

9. Prove the law of quadratic reciprocity: \( q \) is a quadratic residue modulo \( p \) iff \( p_s \) is a quadratic residue modulo \( q \).