(1) Show that $\sqrt{3} + \sqrt{7}/2$ is an algebraic integer, but $\sqrt{3} + \sqrt{5}/2$ is not.

(2) Prove that the rings of integers of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{-6})$ are $\mathbb{Z}[1+\sqrt{5}]$ and $\mathbb{Z}[\sqrt{-6}]$ respectively.

(3) Find an integral basis for the ring of integers in $\mathbb{Q}(\sqrt{3}, \sqrt{2})$.

(4) Let $I$ be the ideal $(2, 1 + \sqrt{-3})$ in $\mathbb{Z}[\sqrt{-3}]$. Prove that $I^2 = 2I$ but $I \neq (2)$. Conclude that ideals in $\mathbb{Z}[\sqrt{-3}]$ do not factor uniquely into prime ideals, and thus $\mathbb{Z}[\sqrt{-3}]$ is not the ring of algebraic integers in $\mathbb{Q}(\sqrt{-3})$.

(5) By considering the factorization of 6, prove that $\mathbb{Z}[\sqrt{-6}]$ is not a unique factorization domain, and factor (6) into prime ideals.

(6) Let $K = \mathbb{Q}(\sqrt{2})$. Prove (using the trace pairing or otherwise) that the ring of integers of $K$ is $\mathbb{Z}[\sqrt{2}]$.

(7) You may assume that the ring of integers in $\mathbb{Q}(\sqrt{-14})$ is $\mathbb{Z}[\sqrt{-14}]$. Let $I = (3, \sqrt{-14} - 1)$ be an ideal in $\mathbb{Z}[\sqrt{-14}]$. Prove that $I$, $I^2$, $I^3$ are not principal, while $I^4$ is.

(8) Let $R$ be a domain, and $K$ be its fraction field. We say that $x \in K$ is integral over $R$ if the ring $R[x]$ is finite as an $R$-module. Let $S$ denote the set of all elements $x \in K$ which are integral over $R$. Prove that $S$ is a ring. We say that $S$ is the integral closure of $R$, and that $R$ is integrally closed if $R = S$.

(9) We say that a domain $R$ is of dimension one if every prime ideal in $R$ is maximal.

Let $R$ be a domain with fraction field $K$, and define a fractional ideal of $K$ to be any $R$-submodule $M \subset K$ such that there exists $r \in R$ with $r \neq 0$ such that $rM \subset R$. Let $I_K$ denote the set of non-zero fractional ideals. Prove that the following are equivalent:

(a) $R$ is Noetherian, integrally closed, and of dimension one.

(b) $I_K$ forms a group under multiplication.

We say that a ring $R$ satisfying the above is a Dedekind Domain.