The First Cosmological Models

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1 Introduction

Einstein’s Equation, including the cosmological constant:

\[ G_{ab} + \Lambda g_{ab} = T_{ab} \]  \hspace{1cm} (1.1)

Now that we have an equation that describes the relationship between spacetime and the energy contained in it, we would like to describe the large-scale geometry of our universe. How do we go about such an arduous task? We will require two other ingredients besides Einstein’s equations to put together such a cosmological model: The first is known as the cosmological principle, and the second is Weyl’s postulate. This report will demonstrate some of the mathematics required, as well as attempt to emphasize the ideas behind the math. After reviewing how we can use symmetries in a space-time to give us a information about the relevant metric, we will solve Einstein’s equation to find the equation of motion of the universe. Two of the first solutions published will be examined, one by Einstein (1917) and the other by de Sitter (1917). These cases are interesting historically and philosophically, as well shall see.

2 Symmetries

2.1 Killing Vectors

Consider a two dimensional sphere, with metric \( ds^2 = d\theta^2 + \sin^2(\theta)d\phi^2 \), embedded in Euclidean 3-space. If we think of a being that lives on this world...

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sphere, he can do three things that will not change the way the landscape looks to him. He can:

1) Walk in one direction;
2) Walk in a direction perpendicular to the direction from 1);
3) He can stand still, and spin around on the spot.

These three independent actions tell us about his living space. The killing vectors that correspond to the above symmetries are just the angular momentum operators in 3-D (or the generators of rotations in SO(3), also known as a basis for the Lie algebra of SO(3)):

\[ \partial_x = -\cos(\phi) \partial_\theta + \cot(\theta) \sin(\phi) \partial_\phi \]  
\[ \partial_y = \sin(\phi) \partial_\theta + \cot(\theta) \cos(\phi) \partial_\phi \]  
\[ \partial_z = \partial_\phi \]  

(If we consider the man standing right on the z-axis, then the two independent ways he can walk are denoted by the vectors \( \partial_x \) and \( \partial_y \), with "spinning on the spot" corresponding to \( \partial_z \).)

Now, let’s try to understand how we can find the Killing vectors for some arbitrary symmetry on a manifold, with an unknown metric \( g_{\mu\nu} \). Following Weinberg (Pg. 375) we say that a metric is form-invariant under a given coordinate transformation \( x \rightarrow x' \) if:

\[ g'_{\mu\nu}(x') = g_{\mu\nu}(x), \forall x. \]  

This says that we can change the coordinates (rotations, translations, etc), and this does not affect the form of the metric, at any point. In general, the metric transforms as a \((0,2)\) tensor:

\[ g_{\nu\mu}(x) = \partial x'^\rho \partial x'^\sigma \frac{\partial x^\rho}{\partial x^\nu} g'_{\rho\sigma}(x') \]  

If the metric is form invariant, we can write this as

\[ g_{\nu\mu}(x) = \partial x'^\rho \partial x'^\sigma \frac{\partial x^\rho}{\partial x^\nu} g_{\rho\sigma}(x') \]  

with the prime missing from the metric on the right side.

Any transformation \( x \rightarrow x' \) that satisfies this property is called an isometry. The above equation generally yields a very complicated relationship between \( x' \) and \( x \), but if we consider an infinitesimal transformation:
\[ x'\nu = x^\nu + \varepsilon \zeta^\nu(x), \varepsilon \ll 1 \quad (2.7) \]

Then, to first order in \( \varepsilon \), using
\[
\frac{\partial x'^\rho}{\partial x^\nu} = \frac{\partial x^\rho}{\partial x^\nu} + \varepsilon \frac{\partial \zeta^\rho(x)}{\partial x^\nu} \quad (2.8)
\]

and
\[
g_{\rho\sigma}(x') = g_{\rho\sigma}(x) + \varepsilon \zeta^\nu(x) \frac{\partial g_{\rho\sigma}(x)}{\partial x^\nu} \quad (2.9)
\]

We get
\[
0 = \frac{\partial \zeta^\rho}{\partial x^\nu} g_{\rho\nu} + \frac{\partial \zeta^\sigma}{\partial x^\nu} g_{\mu\sigma} + \zeta^\nu \frac{\partial g_{\rho\mu}}{\partial x^\nu} \quad (2.10)
\]

This may be rewritten using \( \zeta_\alpha = g_{\rho\alpha} \zeta^\rho \) as
\[
0 = \frac{\partial \zeta^\sigma}{\partial x^\rho} + \frac{\partial \zeta^\rho}{\partial x^\sigma} + \zeta^\nu \left( \frac{\partial g_{\rho\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\rho} - \frac{\partial g_{\rho\nu}}{\partial x^\sigma} \right) \quad (2.11)
\]

using the definition of the Christoffel symbol:
\[
0 = \frac{\partial \zeta_\sigma}{\partial x^\rho} + \frac{\partial \zeta_\rho}{\partial x^\sigma} - 2 \zeta^\nu \Gamma^\nu_{\rho\sigma} \quad (2.12)
\]

We recognize the invariant derivative (covariant derivative), allowing us to arrive at the Killing condition:
\[
0 = \nabla_\rho \zeta_\sigma + \nabla_\sigma \zeta_\rho \quad (2.13)
\]

Solving this equation will yield the Killing vectors \( \zeta_\sigma(x) \) for the given metric. It should be noted that this equation holds at every point on the manifold, but does not (immediately) tell use the entire vector field \( \zeta^\nu(x) \). The entire field can be found using the Riemann curvature tensor to construct a relation between all the derivatives of \( \zeta_\sigma(x) \), thus the taylor series can be reconstructed. I refer you to Weinberg Pg. 377 if you are interested in the details. It should also be noted that this equation depends on the metric (through \( \nabla_\rho \)), so at first glance it is hard to see how this will facilitate finding the metric. The meaning of the Killing vectors and the isometries they create can be thought of as follows: If we existed in this space, there would be no way to tell from the geometry of the space that we have moved at all, had we moved along a Killing vector. Said in another way, "moving" along an isometry (along a Killing vector) is equivalent to "standing still".
2.2 Lie Derivatives

Another interesting and related concept is the Lie derivative. We can define a congruence (a set) of curves $x^a(\lambda)$, parameterized by $\lambda$ on the manifold such that only one curve goes through each point on the manifold. Then, we can naturally define a vector field $dx^a/d\lambda$ on the manifold. If we imagine that we have an arbitrary tensor defined on the manifold at $x = P$, we can imagine "dragging" the tensor along one of the above curves to some point $x = Q$, such that

$$T_{b_1b_2\ldots}^{a_1a_2\ldots}(x = P) \rightarrow T_{b_1b_2\ldots}^{d_1a_2\ldots}(x = Q) \quad (2.14)$$

We can then compare the tensor that is already there, $T_{b_1b_2\ldots}^{a_1a_2\ldots}(x = Q)$, by subtracting the two tensors (as they are of the same type). In the limit as $Q \rightarrow P$ we have what is called a Lie derivative, and it tells us about how tensors change in the direction that the derivative is taken. It should be noted that the Lie derivative is defined with respect to some vector $dx^a/d\lambda$. In the case that this vector is Killing, we expect from the above definition of the Killing field that the Lie derivative will be zero along a Killing vector, which can be shown.

2.3 Isotropic and Homogeneous Spaces

Loosely speaking, a metric space is said to be homogeneous if every "place" "looks like" every other "place". More precisely,

"A metric space is said to be homogeneous if there exist infinitesimal isometries (2.7) that carry any given point $x$ to any other point in its immediate neighborhood. That is, the metric must admit Killing vectors that at any given point take on all possible values."

For our purposes, we will be concerned with "spatially" homogeneous manifolds, i.e. the 3-D space-like hyper-planes in the manifold. A metric space is said to be isotropic if we cannot distinguish any one particular direction from any other, that no matter which way we look, we see the same "landscape". More precisely,

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1 Wald Pg. 92
2 Weinberg, Pg. 378
"A metric is said to be (spatially) isotropic at each point if there exists a congruence of time-like curves (observers), with tangents denoted $u^a$, filling the space-time satisfying the following: Given any point $P$ and any two 'spatial' tangent vectors $s_1^a, s_2^a \in V_p$ (the tangent space), $s_1^a, s_2^a$ orthogonal to $u^a$, there exists an isometry of $g_{ab}$ which leave $p$ and $u^a$ fixed but rotate $s_1^a$ into $s_2^a$.

That the universe is approximately homogeneous over cosmological scales is known as the cosmological principle. As an interesting side note, Weinberg has shown (Weinberg, Pg. 378-379) that any space that is isotropic at every point implies homogeneity. If you cannot distinguish any preferred direction at any point on the manifold, then by moving in any of those directions a small amount the space looks the same. More simply, if space is the "same" everywhere, then we cannot distinguish a preferred direction anywhere in the space, as that would mean that space is different, contrary to homogeneity.

The 2-sphere used above is both homogeneous, as every point looks like every other point, and isotropic, as we can not distinguish between different directions. If we consider a 2-D infinite cylinder embedded in Euclidean 3-space, it will be homogeneous, but not isotropic, as we could distinguish between different directions. (In particular, the cylinder will have 2 Killing vectors, corresponding to moving "up" and "down" along the cylinder, and moving "around" the cylinder. Rotations about one point will not be a Killing vector, as it was for the 2-sphere).

So, how do we put all of the above symmetries to use? We would like to derive a general form of the metric that we can put into Einstein’s equations to model our universe.

3 Spaces of Constant Curvature

3.1 The Metric

In a general $N$ dimensional manifold with metric $g_{ab}$, the maximum number of independent Killing vectors is $^4 N(N + 1)/2$. It can be shown that maximally symmetric spaces (spaces that allow the maximum number of independent Killing vectors) are unique, which means they only differ by some

\footnote{Wald}

\footnote{In $N$ dimensional space, there are $N$ translations and "$N$ choose ($N$-2)" rotations.}

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coordinate transformation. It can also be shown that these spaces have constant curvature, which is evident from the fact that the space is isotropic (or equivalently homogeneous), and can be demonstrated rigorously using Killing vectors (see Weinberg Chapter 13). It can also be shown that the Riemann tensor for a space of constant curvature can be written in the form:

\[ R_{\lambda\mu\nu\sigma} = K(g_{\nu\mu}g_{\lambda\sigma} - g_{\sigma\mu}g_{\lambda\nu}) \]  

where \( K \) is a constant relating to the curvature of the space.\(^5\).

In order to describe spaces of constant curvature, we can define a co-moving, synchronous coordinate system as follows: Pick a 3-D spatially homogeneous hypersurface at time \( t_I \). Assign to it any linearly independent coordinate system \( x^i, i = (1, 2, 3) \). In order to extend this coordinate system into the entire 4-D space-time, we imagine that the universe is full of "dust", and that in one reference frame this dust is spatially stationary everywhere. In such a frame the 4-velocity of each dust particle is:

\[ v^\alpha = (1, 0, 0, 0) \]  

such that

\[ v^\alpha v_\alpha = g_{ab}v^a v^b = -1 \]  

As time changes, these dust particles remain stationary with respect to the spatial coordinates (although they may move with respect to each other)\(^6\). The "dust" was introduced mainly to help imagine such a space, but it is not necessary to create such a coordinate system: we can create the same coordinate system in empty space (I imagine laying a grid in the spatially homogeneous space and placing a "dust" particle at each intersection of my grid lines). Such a coordinate system is called "Gaussian normal coordinates".

In summary, the spatial coordinates move with the "dust", and the time coordinate is the proper time along the "dust" world lines. In order to reconcile the discrete nature of the dust, we can imagine taking the continuum limit as the "dust" goes to a "perfect fluid". From the above construction, the world line of every point ("dust" particle, if you like) is orthogonal to all

\(^5\)Weinberg section 13.2  
\(^6\)If we placed a ruler between two dust particles and allowed it to evolve (stretch or contract) with the space, it would still measure the distance between the dust particles as one meter. The evidence that the space is expanding/contracting would be a red/blue shift of light. This is a little strange when you think about it.
of the spatial directions:

\[
\left( \frac{\partial}{\partial t} \right) \cdot \left( \frac{\partial}{\partial x^i} \right) = 0, \ i = 1, 2, 3 \tag{3.4}
\]

This condition, together with (3.2) imply that the line element is of the form:

\[
ds^2 = -dt^2 + h_{ij}(t, x^k)dx^i dx^j \tag{3.5}
\]

If we know the form of \( h_{ij}(t, x^k) \), then we can describe the space for all time.

We can rationalize this form of \( g_{ab} \) as follows: If we image a triangle in the homogeneous 3-space formed by three of the "dust" particles, at some later time they will still form a triangle. By isotropy, these triangles must be geometrically similar, and by homogeneity the scaling of the triangles must be independent of position. So, this means that time can only enter into the metric by some scaling function (times \( h_{ab} \)) so that the ratios of the distances corresponding to small displacements be the same at all times\(^7\).

Now that we have a general idea of what the form of the metric should be, we will state without proof that the most general line element for the 3-D subspace of homogeneous, isotropic hypersurfaces is of the form:

\[
h_{ij}(t, x^k)dx^i dx^j \equiv [S(t)]^2 d\sigma^2 = [S(t)]^2(e^\lambda dR^2 + R^2(d\theta^2 + sin^2(\theta)d\phi^2)) \tag{3.6}
\]

Where \( \lambda = \lambda(R) \), and the term \( S(t) \) is the scaling factor. The function \( e^\lambda \) is introduced to account for the (constant) curvature of the metric (otherwise we would be in regular flat space). Note that (3.6) is spherically symmetric about every point\(^8\). We will use the condition (3.1) in order to find the form of \( e^\lambda \) in terms of the constant \( K \). First, we find the non-vanishing components Christoffel symbol from the metric, and from those find the Riemann tensor, and then contracting we find the Ricci tensor. Comparing this result with the Ricci tensor found from contracting (3.1) we find that \( e^{-\lambda} = 1 - KR^2 \).

We have arrived at the Robertson-Walker (R-W) Metric\(^9\):

\[
ds^2 = -dt^2 + (S(t))^2(1 + 1/4Kr^2)^{-2}(dr^2 + r^2(d\theta^2 + sin^2(\theta)d\phi^2)) \tag{3.7}
\]

A metric is said to be conformally flat if \( g_{ab} = \Omega^2\eta_{ab} \), where \( \Omega(x) \) is a non-zero differentiable function and \( \eta_{ij} \) is flat. Note that the spatial part

\(^7\)D’Inverno Pg. 316-317

\(^8\)We can always define new variables \( R', \theta', \phi' \) that will leave the metric form unaltered

\(^9\)After introducing the new radial parameter \( R = r/(1 + 1/4K(t)r^2) \)
of the R-W metric is then conformally flat. A result of this is that angles between vectors and ratio’s of magnitudes of vectors are the same for each metric $h_{ij}$ and $\eta_{ij}$, where in this case $\eta_{ij}$ is a Riemannian metric (signature $= + 3$).

We can re-scale this metric using the following:

$$ r \rightarrow |K|^{-1/2} r $$

$$ K \rightarrow |K| \ k $$

$$ S(t) \rightarrow |K|^{1/2} a(t), \ if \ K \neq 0 $$

$$ S(t) \rightarrow a(t), \ if \ K = 0 $$

To get:

$$ ds^2 = -dt^2 + (a(t))^2 \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \right) $$

Where $k = +1, -1$ or 0.

### 3.2 Geometry of Homogeneous Hypersurfaces

Let’s consider the implications of choosing different values of $k$ on the spatial line element $d\sigma^2$.

#### 3.2.1 Case 1: $k = 0$

In this case, the metric on the 3-D hypersurface is:

$$ d\sigma^2 = (a(t_0))^2(d\chi^2 + \chi^2(d\theta^2 + \sin^2(\theta)d\phi^2)) $$

Which we recognize as being Euclidean 3-space. Imagine this: If many people are at the origin, and begin walking radially away from each other while holding hands, the distance that they have to stretch there hands to keep holding each other increases linearly with the distance that they have walked. In math, this means that in flat space the circumference of a circle is proportional to its diameter, and that the area of the great circles that they enclose is proportional to the radius squared. Compare this to the other cases to follow. Also, note that in this case the space is open, or
infinite.

It is interesting to note that the topology of the space is not necessarily unique. The requirement that the space is homogeneous restricts the geometry of the space, but other topological choices are possible. In this case, we can imagine a box of sides $L$, and impose periodic boundary conditions to create a topological 3-Torus. This is analogous to creating a donut out of a sheet of paper by matching the two opposite edges together (2-D torus). In the case of the 3-torus, the volume is no longer infinite\(^{10}\).

### 3.2.2 Case 2: $k = 1$

In this case the coefficient of $dr^2$ is singular when $r = 1$. Define a new coordinate

$$r = \sin(\chi) \quad (3.14)$$

the metric on the 3-D hypersurface, at time $t_0$ is then:

$$d\sigma^2 = (a(t_0))^2(d\chi^2 + \sin^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2))) \quad (3.15)$$

This surface can be embedded in a four dimensional Euclidean space, in which it can be regarded as a 3-sphere. Consider the situation where many people walk away from the origin holding hands again: In this case, the circumference of the circle formed by holding hands is not directly proportional to the distance they have walked. Rather, as they start walking, the circumference increases as $\sin(\chi)$, and the area they enclose goes as $\sin^2(\chi)$. If they keep walking far enough, eventually the area inside their hands goes back to zero (!). This universe is closed, and has a finite volume.

### 3.2.3 Case 3: $k = -1$

In this final case, we can introduce the following coordinate:

$$r = \sinh(\chi) \quad (3.16)$$

the metric on the 3-D hypersurface, at time $t_0$ is then:

$$d\sigma^2 = (a(t_0))^2(d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2))) \quad (3.17)$$

\[^{10}\text{In this flat universe, a spaceship could conceivably fly off in one direction and return from the other direction.}\]
This surface cannot be embedded in Euclidean space (due to the negative sign in the line element), but we can embed it in 4-D Minkowski space with signature +2. Using the following change of variables,

\[
\begin{align*}
    w &= a(t_0) \cosh(\chi), \\
    x &= a(t_0) \sinh(\chi) \sin(\theta) \cos(\phi), \\
    y &= a(t_0) \sinh(\chi) \sin(\theta) \sin(\phi), \\
    z &= a(t_0) \sinh(\chi) \cos(\theta)
\end{align*}
\]

We arrive at the relation between \(w, x, y, z\) to be

\[-w^2 + x^2 + y^2 + z^2 = (a(t_0))^2\]  

(3.22)

Which is the equation for a 3-D hyperboloid in 4-D Minkowski space. In this case, as we join hands and walk away from the origin, the area we enclose goes like \(\sinh^2(\chi)\), which is basically exponential growth for large \(\chi\) (exponential growth rather than quadratic growth, as in the \(k = 0\) case). This space is unbounded and has an infinite volume.

4 **Enter Einstein**

Now that we have the geometry we sought after, we will use Einstein’s equations to show two possible models of the cosmos. But first we need to approximate the matter in the universe.

4.1 **Perfect Fluids**

*Weyl’s Postulate*, the last assumption we need in order to do some cosmology, essentially says that the matter contained in the universe acts as a perfect fluid, as alluded to in section 3.1. To say that a fluid is perfect means that it is has zero viscosity, which means that all of the cross terms in the stress tensor \(T_{ab}\) are zero, and the only interaction in the fluid is through the gravitational interaction. More precisely, ”the ’particles’ in the universe lie in space-time

\footnote{The author is not really sure about the geometric meaning of this (compare a 3-sphere in 4-D Euclidean space), but is comfortable using it as a mathematical tool}
on a congruence of time-like geodesics diverging from a point in the finite (or infinite) past". The stress-energy tensor for a perfect fluid is:

$$T_{ab} = (\rho + p)u_a u_b - pg_{ab}$$

(4.1)

Where $p$ is the pressure which includes random motion of stars and galaxies, heat motion of molecules, radiation pressure, etc. and $\rho$ is the energy density of the universe. This interpretation can be seen by calculating $\nabla_a T_{ab}$.

### 4.2 Solutions to Einstein’s Equations

Once Einstein established his theory of gravity, general relativity (1915), the stage was set to apply it to all sorts of interesting physical situations, including the large scale structure of the universe. It was a generally held belief at the time that the universe obeyed the two previously stated principles, namely the cosmological principle and Weyl’s postulate. It was also held by most at he time, including Einstein, that the universe was static.

#### 4.2.1 Mach’s Principle

In the Newtonian theory of gravity (and all Newtonian dynamics, for that matter) there is a presupposed absolute frame of reference. This leads to the introduction of "fictitious" inertial forces such as the the centrifugal force or the Coriolis force. But, these forces are not fictitious in the sense that jet pilots do pass out if they pull out of a dive to quickly. Can we merely attribute these effects to frame of reference effects, as Newton did? One might ask, what is the physical origin of these forces? Is there a connection between objects in different frames of reference that lead to these forces? Ernst Mach attempted to provide such an answer. Mach’s principle stems from the idea that we can only speak of relative motion, and that the concept of absolute motion is non-sense (contrary to Newton’s view). For a universe with only one particle in it, Mach says we cannot speak of its motion, nor it’s inertial forces. If we consider a universe with many particles we can then speak of inertial forces. So, the presence of other bodies must somehow contribute to the inertial effects. Mach’s answer is that the only inertial frame is that which is fixed relative to the mean motion of ALL of the mass in the universe, which happens to be determined by the "fixed" background stars, as that is

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12D’Inverno Pg. 315
where the majority of the mass exists. To contrast Newton’s view of space and Mach’s, consider the following: If we hang a bucket of water from a rope, twist the rope up, and let go, the bucket will spin. At first, the water will not spin, but after some short time the water will spin with the bucket, causing the surface to become concave. This is due to the fictitious centrifugal force. Now, imagine keeping the bucket stationary (so that the surface is perfectly flat) and then proceed to rotate the fixed background stars. In Newton’s view, the water will remain perfectly flat, but in Mach’s opinion water will become concave and begin to spin in the bucket.

Consider the case when the background stars have a non-isotropic mass distribution. In Mach’s view this would imply that the inertial effects due to the stars would be non-isotropic. Hughes and Drever (1960) established in an experiment that the average mass is isotropic to one part in $10^{18}$, meaning that either Mach’s principle is incorrect, or the universe is isotropic. A consequence of Mach’s principle is that the matter distribution determines the geometry, and that if there is no matter, then there is no geometry, where geometry it is referring to the form of force free particle motion in the universe. These are the constraints that Einstein adopted when he first set out to model the cosmos.

### 4.2.2 Einstein 1917: A Static Closed Universe

When considering the possible homogeneous solutions to the field equations, Einstein, along with the mathematician Jacob Grommer, tried solutions that were degenerate at infinity and found them “unsatisfactory”, in accordance with his Machian prejudice. The problem with an open universe, just as in any boundary value problem, is that the boundary influences the solution, typically through the constants of integration. If these constants are fixed, then when we change coordinates they will take different values, thus conflicting with the basic tenant of relativity, that physics should be the same in all coordinate systems.\(^{13}\). In order to reconcile his difficulties, he postulated a spatially closed universe.

Starting with the R-W metric with , we wish to find the Einstein tensor, and subsequently the governing equations of the universe. There are various shortcuts that we can take in order to reduce the length of the calculation. We can deduce before we begin that the three spatial components will yield

\(^{13}\) A good account of the philosophical issues with boundary conditions in GR can be found in Kerszberg
the same relationship when substituted into Einstein’s equation, because of the spatial symmetries on the manifold in question. See Wald for a detailed analysis. The method used here involves using a computer algebra package called ”GRTensorJ” which is available as a java applet on the web.\textsuperscript{14} The results of the calculations can be seen in the appendix. The governing equations are:

\begin{equation}
3 \frac{k + \left( \frac{da(t)}{dt} \right)^2}{(a(t))^2} - \Lambda = 8\pi\rho
\end{equation}

\begin{equation}
\frac{k + \left( \frac{da(t)}{dt} \right)^2 + 2a(t) \frac{d^2a(t)}{dt^2}}{(a(t))^2} - \Lambda = -8\pi\rho
\end{equation}

It can be shown that for \(\Lambda = 0\), \(\frac{da}{dt} = 0\) the pressure and the energy density must satisfy \(\rho = -3p\). It does not make physical sense for either of these quantities to be negative, so in order to rectify this the cosmological constant was introduced.

In the present universe, it is observed that the pressure \(p\) is on the order of \(10^{-15}\) orders of magnitude smaller then the energy density \(\rho\), so as an approximation we will set it to zero in the following sections. Equation 4.3 can be integrated directly (let \(a = a(t)\)):

\begin{equation}
a^2 \left( k + \left( \frac{da}{dt} \right)^2 \right) + 2a \frac{d^2a}{dt^2} - a^2\Lambda = 0
\end{equation}

Multiply by \(\frac{da}{dt}\)

\begin{equation}
\frac{da}{dt} a^2 \left( k + \left( \frac{da}{dt} \right)^2 \right) + 2 \frac{da}{dt} a \frac{d^2a}{dt^2} - \frac{da}{dt} a^2\Lambda = 0
\end{equation}

\begin{equation}
\frac{d}{dt} \left( a \left( \frac{da}{dt} + k \right) - \frac{1}{3}\Lambda a^3 \right) = 0
\end{equation}

Integrating yeilds:

\begin{equation}
a \left( \frac{da}{dt} + k \right) - \frac{1}{3}\Lambda a^3 = C
\end{equation}

\textsuperscript{14}http://grtensor.org/ I encourage interested parties to investigate. It will convert to tex format as well, which is very convenient.
Where \( C \) is a constant of integration. We can use (4.2) to find the constant:

\[
k + \left( \frac{da}{dt} \right)^2 = \frac{(8\pi \rho + \Lambda) a^2}{3}
\]  

(4.8)

So we have

\[
C = \frac{8}{3} \pi a^3 \rho
\]  

(4.9)

Using this result, we can eliminate \( \rho \) in (4.2). The result is

\[
\left( \frac{da}{dt} \right)^2 = \frac{C}{a} + \frac{1}{3} \lambda a^2 - k
\]  

(4.10)

This equation is known as Friedmann’s equation, and it describes the time evolution of the scaling function \( a(t) \) in the metric.

The Einstein solution of 1917 is for a closed, static universe \( (k = +1) \) with the value of \( \lambda \) chosen such that \( da/dt = 0 \). This condition yields:

\[
\Lambda_{\text{critical}} = \frac{3}{(a_0)^2} - 8\pi \rho = \frac{1}{(a_0)^2}
\]  

(4.11)

Where the first part is a result of (4.10) and the second is a result of (4.3), and \( a_0 \) is the radius of the static universe. If \( \Lambda \) is chosen as such, then the value of \( a \) does not change with time, which is to say that the universe is stationary. We can see from (4.11) that the energy density of the universe is related to the radius by

\[
\rho = \frac{1}{4\pi (a_0)^2}
\]  

(4.12)

So, the energy per volume of space in Einstein’s universe is inversely proportional to the surface area of the sphere that encloses the universe. Note that we can re-write Friedmann’s equation in the form of a classical Hamiltonian (with a negative constant energy, the implications of which are not clear, but we will continue for illustrative purposes):

\[
\left( \frac{da}{dt} \right)^2 + V(a) = (\text{constant})
\]  

(4.13)

Where the ”potential” is

\[
V(a) = - \left( \frac{8}{3} \pi \rho + \frac{1}{3} \Lambda \right) a^2
\]  

(4.14)
We see that the "potential" is an inverted parabola, which is zero if \( \Lambda = \Lambda_{\text{critical}} \). This solution is unstable under small perturbations in \( a \), due to the fact that it is at a local maximum in the "potential". What the case be if \( \Lambda = 0 \)? The "potential" becomes

\[
V(a) = -\frac{8}{3} \pi \rho a^2
\]

It is now evident why Einstein introduced the cosmological constant, for without it we would not have a stationary solution in a closed universe, as Einstein desired. Also note that his curved space is a result of the non-zero energy density in the universe, consistent with Machian views.

4.2.3 de Sitter 1917:

If we set \( \rho = p = k = 0, \Lambda > 0 \), then equations 4.2 becomes

\[
3 \left( \frac{\dot{a}(t)}{a(t)} \right)^2 - \Lambda = 0
\]

which can be solved immediately

\[
a(t) = \exp \left( \left( \frac{\Lambda}{3} \right)^{\frac{1}{2}} t \right)
\]

This is an expanding solution with no matter in the universe, contradicting Mach’s principle. It should be noted that the above equations were found originally by de Sitter written as a variant of the Schwarzschild metric. It is possible to derive the same equation from a modified Schwarzschild metric including the cosmological constant, thus showing that space does not need be flat in the absence of matter, in violation of Mach’s principle. At the time, it was the only model of an expanding universe. Many thought that our universe was some kind of combination of Einstein’s "matter without motion" universe and de Sitter’s "motion without matter" universe\(^{15}\).

5 Einstein vs. de Sitter

Little mention has been made of the history behind the mentioned cosmological models. Einstein, while arriving at his solution was primarily

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concerned with retaining Machian principles. He was decidedly opposed to de Sitter’s solution as a physical one, arguing that ”the field $g_{ab}$ ought to be conditioned by matter, otherwise it would not exist at all”\textsuperscript{16}. de Sitter was firmly opposed to Einstein’s Machian ideas, and thought that the notion of fixed background stars influencing the inertia of objects was contrary to the notion of relativity. de Sitter had philosophical problems with considering things outside of our neighborhood (outside of our past light cone) as physical objects, when we have no way of knowing they exist. The debate evolved for years, Einstein eventually abandoning the Machian viewpoint in favour of the view that spacetime and the matter in it were inexorably linked and that they could not be thought of as separate entities. In 1917, Einstein held the Machian position that the matter completely influenced the metric, and the metric did not influence matter. Einstein (1920) changed his views, and emphasized the metric field as being a medium ”which is itself devoid of all mechanical and kinematical qualities, but helps to determine mechanical (and electromagnetic) events”.

In 1929 Edwin Hubble published his findings on the galactic redshift, providing evidence that the universe is expanding, thus discarding the original motivation for introducing the cosmological constant. It was then that Einstein remarked that the constant was his ”greatest blunder”.

6 Conclusions

Starting from the Einstein equations and introducing the concepts of symmetries in spacetime, the origin of the Robertson - Walker metric was demonstrated. Two of the earliest models of the cosmos were discussed, along with some of the philosophical biases held. The author has striven to convey a mixture of the mathematics involved, as well as the intuitive (or not so intuitive) ideas behind the mathematics. It should be noted that much of the debate between Einstein and de Sitter has been omitted for reasons of length. As well, it should be noted that approximately 15 years after these events, Einstein and de Sitter collaborated to produce the “Einstein - de Sitter” model of the universe, as they are documented today, which have nothing to do with the above models.

\textsuperscript{16}Kerszberg
7 References


8 Appendix

First, the R-W metric:

\[ g_{tt} = -1, \quad g_{rr} = \frac{a^2}{1 - kr^2}, \quad (8.1) \]

\[ g_{\theta\theta} = a^2 r^2, \quad g_{\phi\phi} = a^2 r^2 \sin^2(\theta) \quad (8.2) \]

Now, the Christoffel symbols:

\[ \Gamma^r_{rr} = -\frac{kr}{1 - kr^2} \quad (8.3) \]

\[ \Gamma^t_{rr} = \frac{a(t) \frac{da(t)}{dt}}{1 - kr^2} \quad (8.4) \]

\[ \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = r^{-1} \quad (8.5) \]

\[ \Gamma^r_{rt} = \Gamma^\theta_{rt} = \Gamma^\phi_{rt} = \frac{d}{dt} a(t) \quad (8.6) \]

\[ \Gamma^r_{\theta\theta} = (-1 + kr^2)r \quad (8.7) \]

\[ \Gamma^r_{\phi\phi} = ar^2 \frac{d}{dt} a(t) \quad (8.8) \]
\[ \Gamma_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)} \]  

(8.9)

\[ \Gamma_{\phi\phi}^r = (-1 + kr^2)r(\sin(\theta))^2 \]  

(8.10)

\[ \Gamma_{\phi\phi}^\theta = -\sin(\theta)\cos(\theta) \]  

(8.11)

\[ \Gamma_{\phi\phi}^r = a(t)r^2(\sin(\theta))^2 \frac{d}{dt}a(t) \]  

(8.12)

Next, the Riemann tensor:

\[ R_{\theta\phi\theta}^r = r^2 \left( k + \left( \frac{d}{dt}a(t) \right)^2 \right) \]  

(8.13)

\[ R_{\phi\theta\phi}^r = r^2 (\sin(\theta))^2 \left( k + \left( \frac{d}{dt}a(t) \right)^2 \right) \]  

(8.14)

\[ R_{\theta\phi\theta} = R_{\phi\theta\phi} = R_{\psi\phi\psi} = \frac{\frac{d}{dt}a(t)}{a(t)} \]  

(8.15)

\[ R_{\phi\theta\phi}^\theta = r^2 (\sin(\theta))^2 \left( k + \left( \frac{d}{dt}a(t) \right)^2 \right) \]  

(8.16)

\[ R_{\phi\theta\phi}^\theta = -r^2 \left( k + \left( \frac{d}{dt}a(t) \right)^2 \right) \]  

(8.17)

\[ R_{\theta\psi\psi} = -a(t) \frac{d^2}{dt^2}a(t) \]  

(8.18)

\[ R_{\theta\psi\psi} = a(t) \frac{d^2}{dt^2}a(t) - 1 + kr^2 \]  

(8.19)

\[ R_{\phi\theta\phi} = -a(t)r^2 (\sin(\theta))^2 \frac{d^2}{dt^2}a(t) \]  

(8.20)

Next Ricci:

\[ R_{rr} = -\frac{2k + 2 \left( \frac{d}{dt}a(t) \right)^2 + a(t) \frac{d^2}{dt^2}a(t)}{-1 + kr^2} \]  

(8.22)
\[ R_{\theta\theta} = 2 kr^2 + 2 r^2 \left( \frac{d}{dt}a(t) \right)^2 + a(t)r^2 \frac{d^2}{dt^2}a(t) \] (8.23)

\[ R_{\phi\phi} = 2 kr^2 (\sin(\theta))^2 + 2 r^2 (\sin(\theta))^2 \left( \frac{d}{dt}a(t) \right)^2 + a(t)r^2 (\sin(\theta))^2 \frac{d^2}{dt^2}a(t) \] (8.24)

\[ R_{tt} = -3 \frac{d^2}{dt^2}a(t) \] (8.25)

And finally the Einstein tensor:

\[ G_{rr} = k + \left( \frac{d}{dt}a(t) \right)^2 + 2a(t)\frac{d^2}{dt^2}a(t) \left( \frac{1}{1 + kr^2} \right) \] (8.26)

\[ G_{\theta\theta} = -kr^2 - r^2 \left( \frac{d}{dt}a(t) \right)^2 - 2a(t)r^2 \frac{d^2}{dt^2}a(t) \] (8.27)

\[ G_{\phi\phi} = -kr^2 (\sin(\theta))^2 - r^2 (\sin(\theta))^2 \left( \frac{d}{dt}a(t) \right)^2 - 2a(t)r^2 (\sin(\theta))^2 \frac{d^2}{dt^2}a(t) \] (8.28)

\[ G_{tt} = 3 k + \left( \frac{d}{dt}a(t) \right)^2 \] (8.29)

The components of the stress-energy tensor are

\[ T_{rr} = p \frac{(a(t))^2}{1 + kr^2} \] (8.30)

\[ T_{\theta\theta} = -p(a(t))^2 r^2 \] (8.31)

\[ T_{\phi\phi} = -p(a(t))^2 r^2 (\sin(\theta))^2 \] (8.32)

\[ T_{tt} = \rho \] (8.33)

So we have 4 equations. Evidently, the \( \theta\theta \), the \( \phi\phi \) and the \( rr \) equations are identical, as expected. The two independent equations are:
\[
3 \frac{k + \left( \frac{d}{dt} a(t) \right)^2}{(a(t))^2} - \Lambda = 8\pi \rho 
\]  
(8.34)

\[
\frac{k + \left( \frac{d}{dt} a(t) \right)^2 + 2 a(t) \frac{d^2}{dt^2} a(t)}{(a(t))^2} - \Lambda = -8\pi p 
\]  
(8.35)

This completes the long and drawn out process of finding Einstein’s equations from a given metric and stress-energy tensor.