On the Positive Energy Theorem

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Introduction

In an isolated physical system, one would expect (physically) the total energy of that system to be positive. In the general relativity setting, for example, we may naively suspect that the total energy of some space-like hypersurface (a choice of space for a fixed time) to be simply the sum of the energy associated to the matter field(s) and the energy associated to the gravitational field. And surely in the absence of such fields the total energy should be zero.

The situation, however, is complicated by the coupling of the matter fields and gravitational fields as prescribed by Eistein’s field equations in general relativity: the energy associated to the matter field and the energy associated to the gravitational field cannot be defined separately. Nevertheless, the Positive Energy Theorem, first proved by Schoen and Yau [1], and later by Witten [2], reassures our intuition through mathematics. The result and its proofs are non-trivial, and as we shall see, the non-triviality starts with the mathematical formulation of the intuitively simple problem. In this paper, we study the mathematical formulation of the problem, and some general aspects of Schoen and Yau’s proof as outlined in a paper by J. Kazdan [3].

The four main ingredients required to understand the mathematical formulation of the Positive Energy Theorem are as follows.

1. Eistein’s field equations for spacetime \((N, \gamma)\)
2. An asymptotically Euclidean spacelike hypersurface \((M, g)\) isometrically immersed in \((N, \gamma)\)
3. The dominant energy condition for the energy-momentum tensor \(T_{\mu\nu}\)
4. The definitions of total energy and total momentum in this setting.

We will now begin to look at these ingredients, and how they combine to give a precise meaning to our intuitive expectations regarding the positive energy of a space-like hypersurface in spacetime.
1 The Setting

Let \( N \) be a four dimensional spacetime with Lorenztian metric \( \gamma \) that satisfies Einstein’s field equation:

\[
\bar{R}_{\mu\nu} - \frac{1}{2} \bar{R} \gamma_{\mu\nu} = 8\pi G T_{\mu\nu}
\]

where \( \bar{R}_{\mu\nu} \) is the Ricci curvature tensor, \( \bar{R} \) is the scalar curvature of \( (N, \gamma) \), \( T_{\mu\nu} \) is the energy-momentum tensor and \( G \) is a physical constant. An important point to observe, of course, is that the gravitational field, encoded in the metric \( \gamma_{\mu\nu} \) is related to the matter fields encoded in the energy momentum tensor \( T_{\mu\nu} \). And so the gravitational field and matter fields are coupled by the above equation.

We will be interested in a (complete, oriented) imbedded space-like hyper-surface \((M, g) \hookrightarrow (N, \gamma)\), which is asymptotically Euclidean. Intuitively, this means that outside some bounded region the metric \( g \) of \( M \) (induced from the metric \( \gamma \) of \( N \)) approaches the flat metric of Euclidean space, the further we depart from the bounded region. More precisely, we say \((M, g)\) is asymptotically Euclidean if for some compact set \( K \subset M \), the compliment \( M - K \) can be written as a finite union

\[
M - K = \bigcup_{k=1}^{n} M_k
\]

where each \( M_k \) is diffeomorphic to the compliment of a ball in \( \mathbb{R}^3 \). The sets \( M_k \) are called the ends of \( M \). Moreover, we require that under these diffeomorphisms, the components of the metric in \( M_k \subset M \) (in Euclidean coordinates) should be of the form

\[
g_{ij} = \delta_{ij} + a_{ij}
\]

where \( a_{ij} \to 0 \), as \( ||x|| \to \infty \). There are more precise conditions that describe what we have stated, but we will not be requiring that level of detail. The interested reader can consult Schoen-Yau [1]. (For a brief discussion of the second fundamental form associated to the immersion \( M \hookrightarrow N \), the reader may wish to review the appendix.)

The asymptotic flatness condition on \( M \) may be viewed as a boundary condition imposed on the spacetime \( N \). Physically, its interpretation is that the system is isolated [4].

We will now impose a condition on the energy-momentum tensor \( T_{\mu\nu} \), commonly called the dominant energy condition. The condition is considered physically reasonable, as most fields satisfy the condition [4]. \( T_{\mu\nu} \) satisfies the dominant energy condition if

\[
i \text{ for every non-spacelike vector } \eta^\mu, T_{\mu\nu} \eta^\mu \eta^\nu \geq 0, \text{ and }
\]

\[
ii \ T_{\mu\nu} \eta^\mu \text{ is also non-spacelike.}
\]

Physically, the first inequality demands that the local energy density is non-negative [5].
We now interpret the second condition using Einstein’s field equation. Now, our hypersurface has codimension 1. So we may define on $M$ a future pointing unit normal by simply completing a basis for $T_pN$. That is, we may write $T_pN = T_pM \ominus T_pM^{\bot}$ and since $M$ is spacelike, we may define a future pointing unit normal $\eta$ for a point $p \in M$ to be a unit vector in $T_pM^{\bot}$. So if we let $\{\eta, e_1, e_2, e_3\}$ be an orthonormal Lorentz frame at $p \in N$, then the dominant energy condition states that $T_{\mu\nu}\eta^\mu = T^0_\nu$ is non-spacelike. That is,

$$T^2_{00} \geq T^2_{01} + T^2_{02} + T^2_{03}.$$ 

Physically, one interprets $\mu = T^0_0$ as the local energy density and $J^i = T^0_i$ as the local angular momentum [4]. The conditions may be re-expressed as (i) $\mu \geq 0$, and (ii) $\mu \geq \|J\|$. 

Now, we will try to re-express the dominant energy condition in terms of the intrinsic geometry of $(M, g) \hookrightarrow (N, \gamma)$. To do this let’s review some differential geometry. 

Recall that our immersed manifold $(M, g) \hookrightarrow (N, \gamma)$ has the property that the metric on $M$ is the one induced by the metric on $N$. To understand the meaning of this, let’s introduce coordinates $\{y^0, y^1, y^2, y^3\}$ on $N$, and suppose the hypersurface $M$ has coordinates $\{x^1, x^2, x^3\}$. Then we may view our immersion $\theta : M \hookrightarrow N$ as a ‘change of variables’ of $M$ to the image $\theta(M)$. And to say that the metric $g$ on $M$ is induced by the metric $\gamma$ on $N$, we describe this condition by a change of variables formula. That is,

$$\gamma_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} = g_{ij}. \quad (1)$$

This gives us the necessary notation to write the so-called fundamental equations of Gauss (2) and Codazzi (3) given below (see Spivak [6]). These equations relate the Riemann curvature tensor $\bar{R}_{\alpha\beta\gamma\delta}$ associated with $(N, \gamma)$ to the curvature tensor $R_{ijkl}$ associated with $(M, g)$.

$$R_{ijkl} = -h_{ik}h_{jl} + h_{il}h_{kj} + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} \quad (2)$$

$$\nabla_k h_{il} - \nabla_l h_{ik} = R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^k} \frac{\partial y^\gamma}{\partial x^l} \frac{\partial y^\delta}{\partial x^j} \quad (3)$$

where recall $h_{ij}$ is the second fundamental form associated to the imbedding, $\nabla$ denotes covariant differentiation on $M$, and $\eta^\beta$ is a unit normal vector.

Using Gauss’ equation (2), we may find corresponding formulae relating the Ricci curvature tensors, and the scalar curvatures of $M$ and $N$. Let’s do this by applying $g^{ik}$ to Gauss’ equation and recalling the formula (1):

$$R^k_{jkl} = -h^k_l h_{jl} + h^k_l h_{kj} + R_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} \gamma_{\alpha\gamma} \frac{\partial x^i}{\partial x^\mu} \frac{\partial x^j}{\partial x^\nu} \frac{\partial x^k}{\partial x^\lambda} \frac{\partial x^l}{\partial x^\nu} \frac{\partial y^\delta}{\partial y^\gamma}$$

which gives

$$R_{jl} = -h^k_l h_{jl} + h^k_l h_{kj} + R_{\delta\delta} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\delta}{\partial x^l} \eta^\beta. \quad (4)$$
Applying $g^{jl}$ to the above gives

$$R_{jl}g^{jl} = -h_k^k h_l^l + R_{\beta\delta} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\delta}{\partial x^l} \gamma_{j\beta\delta} \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^l}{\partial y^\delta}$$

$$R = - (h_k^k)^2 + h_l^l h_k^k + \bar{R}$$

We will now use equations (4) and (5) and substitute them into Einstein’s field equation (ignoring the constant $8\pi G$) to obtain:

$$\bar{R}_{\beta\delta} - \frac{1}{2} \bar{R} \gamma_{\beta\delta} = T_{\beta\delta}$$

Now, we claim that for our unit normal $\eta$, we must have

$$\frac{\partial x^i}{\partial y^\alpha} \frac{\partial x^l}{\partial y^\delta} \eta^\beta \eta^\delta = 0$$

To see this, we observe that since $\eta^\beta$ is orthogonal to each vector in $T_pM$ then

$$\gamma_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \eta^\beta = 0$$

But then its corresponding 1-form $\eta_\beta$ satisfies

$$\gamma^\alpha_{\beta\delta} \frac{\partial x^i}{\partial y^\alpha} \eta_\beta = 0$$

Therefore,

$$\frac{\partial x^i}{\partial y^\alpha} \eta^\alpha = 0$$

which shows the claim.

So we see that applying equation (6) to $\eta^\beta \eta^\delta$, we obtain (using the above claim):

$$(R_{jl} + h_k^k h_{jl} - h_l^l h_k^k) \frac{\partial x^j}{\partial y^\beta} \frac{\partial x^l}{\partial y^\delta} \eta^\beta \eta^\delta - \frac{1}{2} (R + (h_k^k)^2 - h_l^l h_k^k) \gamma_{\beta\delta} \eta^\beta \eta^\delta = T_{\beta\delta} \eta^\beta \eta^\delta$$

and since $\eta$ is a unit timelike normal, $\gamma_{\beta\delta} \eta^\beta \eta^\delta = -1$. This gives

$$\frac{1}{2} (R + (h_k^k)^2 - h_l^l h_k^k) = T_{00} = \mu$$

This provides a link between the geometry of our hypersurface and the coefficient $\mu$ of the stress-energy tensor. The other link comes from using the Codazzi
equation (3) instead.

\[
(\nabla_k h_{il} - \nabla_l h_{ik})\gamma^{\alpha\gamma} = \bar{R}_{\alpha\beta\gamma\delta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\gamma}{\partial x^k} \frac{\partial y^\delta}{\partial x^l} \eta^{\beta}
\]

Again, we use Einstein’s equation to relate the geometry to the energy momentum tensor.

\[
\nabla_k h^k_l - \nabla_l h^k_k - \frac{1}{2} \bar{R}_{\gamma\beta\delta} \frac{\partial y^\delta}{\partial x^l} \eta^{\beta} = T_{\beta\delta} \frac{\partial y^\delta}{\partial x^l} \eta^{\beta}
\]

Observe that \(\gamma_{\beta\delta} \frac{\partial y^\delta}{\partial x^l} \eta^{\beta} = 0\), and apply \(g_{il}\) to both sides to get

\[
\nabla_k h^{lk} - \nabla_l h^{k} - T_{\beta\delta} \eta^{\beta} \frac{\partial y^\delta}{\partial x^l} \eta^{\beta} = \gamma_{\alpha\delta} (T_{\beta\delta} \eta^{\beta}) \frac{\partial x^l}{\partial y^\alpha}
\]

\[
\nabla_k (h^{lk} - h^l_k) = J^l \tag{8}
\]

Equations (7) and (8) above give us the desired geometric interpretation of the energy condition on the energy-momentum tensor. One can begin to get a feel for the geometry of the situation if we consider the special case where \(h_{ij}\) satisfies \(h_{kk} = 0\). In this case, the condition \(\mu \geq 0\) and equation (7) show that

\[
0 \leq \mu = \frac{1}{2} (R + 0^2 - h^k_k h^k_k)
\]

forces \(R \geq 0\), since \(h^k_k h^k_k \geq 0\) (a sum of squares).

## 2 The Positive Energy Theorem

We are now in a position to state the positive energy theorem, having understood the various constraints of our mathematical description of the physical system (spacetime). But we must first define mathematically the notion of total energy associated to the system.

There are many such definitions, but the approach considered by Schoen-Yau, and Witten considers the asymptotic behaviour, at large distances, of the gravitational field. \[2\] We note first, that one cannot define the total energy of the gravitational field as the integral of some localized energy density associated to the gravitational field. To see why, consider Einstein’s equivalence principle. One can always find a local frame of reference in which all the Christoffel coefficients \(\Gamma^\gamma_{\alpha\beta} = 0\). That is, in which all local effects of gravity disappear. So it doesn’t make sense to attempt to define the total energy in this way, as gravitational energy is non-localizable. \[8\]

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One defines the total energy and total momentum of an asymptotically flat manifold by defining the total energy and total momentum of each end $M_k$ of $M$. Letting $S_{r,k}$ be the 2-sphere of radius $r$ in $M_k$ (note that we are identifying $M_k$ with its diffeomorphic image in $\mathbb{R}^3$) we define the total energy $E_k$ of the end $M_k$ to be

$$E_k = \lim_{r \to \infty} \frac{1}{16\pi G} \int_{S_{r,k}} (\partial_j g_{ij} - \partial_i g_{jj})dS^i$$  \hspace{1cm} (9)$$

and we define the total momentum $p^i_k$ of the end $M_k$ to be

$$p^i_k = \lim_{r \to \infty} \frac{1}{8\pi G} \int_{S_{r,k}} (h_{ij} - \delta_{ij} h_{ll})dS^i$$  \hspace{1cm} (10)$$

where $dS^i$ represents the area element of the 2-sphere, and $g_{ij}$ is the metric on $M$, and $h_{ij}$ is the second fundamental form associated with the imbedding.

The above definition of energy is called the ADM energy. Note that the integrand of (9) is not tensorial (partial derivatives are not tensorial), so it is coordinate dependent. (Note also that repeated indices are summed over, but we don’t raise the index in this case, as the expressions are not tensorial.) However, in taking the limit, it has been shown (see Bartnik [12]) that the above definition is well-defined and does not depend on the choice of coordinates.

The limits above reflects the measurement of energy at spatial infinity. But with an asymptotically flat spacetime, perhaps there are different vantage points of infinity from which to view this energy? Thus it may be necessary to consider the global energy as viewed from each end, as in the definition above. These comments are merely the author’s speculation. (Is it reasonable that the energy for each end be the same? i.e. $E_i = E_j$ for all $i = j$)

The main thing to note is that the above definitions depend only on the metric and the second fundamental form - in other words, solely in terms of the data on $M$. So these definitions are useful if one wishes to consider our spacelike hypersurface as an initial data set which will evolve under Einstein’s field equations.

Physically, the above definition of energy has the interpretation of the total energy including contributions (which cannot be defined seperately) from the matter itself and from the gravitational field. [9]

**Positive Energy Theorem 1** Let $(M,g) \hookrightarrow (N, \gamma)$ be an asymptotically Euclidean space-like hypersurface, where the metric $\gamma$ satisfies Einstein’s field equation $\bar{R}_{\mu\nu} - \frac{1}{2}R_{\mu\nu} = 8\pi GT_{\mu\nu}$, and $T_{\mu\nu}$ satisfies the dominant energy condition. Then for each end $M_k$, $E_k \geq ||p_k||$. Moreover, if $E_k = 0$ for some $k$, then $M$ is flat and has only one end.

Observe that if the 4-vector $(E_k, p^1_k, p^2_k, p^3_k)$ satisfies $E_k \geq ||p_k||$, then it is a timelike vector for the time slice of the given hypersurface. And so it must be for any time. Therefore, if the total energy of the initial data hypersurface $M$ is positive, then it is positive for all time.
Note. The theorem says that the total energy of $M$ can be zero only if the hypersurface is flat. That is, only if it can be imbedded as a hypersurface in Minkowski spacetime with the induced metric and second fundamental form. That is, in this case, $M$ will evolve under Einstein’s field equations to be Minkowski spacetime. [4]

3 About the proof by Schoen and Yau

Next, we turn to some of the aspects of Schoen and Yau’s proof. Specifically, we will discuss some highlights of a special case of their proof, which gives a taste of the complexity of their geometrical argument.

The proof itself is closely related to the problem in Riemannian geometry of finding a metric on a given Riemannian manifold with positive scalar curvature. Schoen and Yau solved some of these geometrical problems using minimal surfaces, and proved the positive energy theorem as a consequence of that work. [3]

A minimal surface is a two dimensional analogue of a geodesic: it is a critical point of the surface area functional $A$. That is, the first variation $\delta A(S)$ of such a surface $S \hookrightarrow M$, with $M$ a three dimensional manifold (such as the space-like hypersurface considered above), is zero. And for $S$ to have minimal area, we require the second variation $\delta^2 A(S) \geq 0$. The relevant formula is the so-called second variation formula (see Lawson [10]), but we will use a modified version of it to describe the stability condition $\delta^2 A(S) \geq 0$. The formula asserts that for all smooth functions $f : S \rightarrow \mathbb{R}$ with compact support,

$$0 \leq \delta^2 A(S) = \int_S (|\nabla_S f|^2 - (\frac{1}{2}(R + \|b\|^2) - K)f^2) dA$$

where $dA$ is the area element on $S$, $R$ is the scalar curvature of $M$, $K$ is the Gaussian curvature of $S$, and $b_{ij}$ is the second fundamental form of the imbedding $S \hookrightarrow M$. [3]

If we suppose that $S$ is compact, then letting $f = 1$, we have

$$\frac{1}{2} \int_S (R + \|b\|^2) dA \leq \int_S K dA$$

Recall the Gauss-Bonnet theorem:

$$\int_S K = 2\pi \chi(S)$$

where $\chi(S)$ is the Euler characteristic of $S$. Then our inequality becomes

$$\frac{1}{2} \int_S (R + \|b\|^2) dA \leq 2\pi \chi(S).$$

Schoen and Yau use this result to show:

**Proposition 1** There is no Riemannian metric for the torus $M = T^3$ having positive scalar curvature $R$. And if $R$ is non-negative, then the metric must be flat, and $R = 0$. [3]
Sketch of proof. Schoen and Yau show that given a metric \( g \) on \( T^3 \), one can find a compact, minimal torus \( T^2 \) imbedded in \( T^3 \). Since the Euler characteristic \( \chi(T^2) = 0 \), the inequality above shows that \( R \) can’t be positive. But \( R \) may be identically zero with Riemann curvature non-zero! But Schoen and Yau show that this yields a new metric \( g' \) on \( T^3 \) with positive scalar curvature \( R' \), which we know can’t happen. Therefore the metric \( g \) really is flat.

This gives the following corollary:

**Corollary 1 (Special case of positive energy theorem)** Given a metric \( g \) on \( \mathbb{R}^3 \) that satisfies \( g_{ij} = \delta_{ij} \) outside some compact set, and with scalar curvature of \( g \) non-negative, we must have that \( g_{ij} = \delta_{ij} \) everywhere.

Proof. The compact set referred to may be placed inside a large cube. And the cube may be continuously deformed to obtain a torus \( T^3 \). So the resulting metric on the torus will have non-negative scalar curvature, since \( g \) had non-negative curvature by hypothesis, and the deformation is a local isometry. By the above result, the metric must be flat inside the torus as well, and so it is flat everywhere.

To see why the above corollary is a special case of the positive energy theorem, consider the condition \( g_{ij} = \delta_{ij} \) outside a compact set as a strong version of asymptotic flatness. And recall that considering the special case \( h_{kk} = 0 \) yields (via the geometric version of the dominant energy condition) \( R \geq 0 \) as discussed at the end of the first section. Then this corollary forces the metric to be the flat metric, and so the total energy is zero, and (by assumption) there is only one end.

In their proof, Schoen and Yau first treat the special case \( h_{kk} = 0 \) as above, only with the more general constraint of asymptotic flatness. In particular, the nice argument using the Gauss-Bonnet theorem will fail twofold, as compactness was necessary to invoke the Gauss-Bonnet theorem as well as in constructing the minimal surface \( S \) on which we applied the Gauss-Bonnet theorem. (We note, nevertheless, that the implication of \( R \geq 0 \) from the assumption \( h_{kk} \) still holds in the more general setting.)

The general idea of the proof of this special case is the same. One still shows the existence of a minimal surface, and uses an alternative argument than Gauss-Bonnet to reach a contradiction. The general case is then proved by reducing to the special case via a kind of smooth deformation of the metric. For a little more detail, yet still a sketch, the reader may wish to consult J. Kazdan’s paper, [3] or a short paper by Yau. [4]

### 4 Some concluding remarks

The importance of the positive energy theorem lies mainly in that it is related to the stability of Minkowski space, [2] and it assures us that the ADM mass/energy has some physical properties we would hope to have from a good definition of mass/energy. Namely that the total mass/energy be positive, unless the spacetime is trivial, in which case the total mass/energy should be zero.
As we mentioned in the remark following the statement of the theorem, the positive energy theorem tells us that the total energy of spacetime is zero only if the initial data set is the trivial one. In other words, only if the spacetime obtained by evolving from the initial data on the spacelike hypersurface according to Einstein’s equations is (flat) Minkowski spacetime.

The positive energy theorem certainly has other implications in general relativity, but they will not be discussed in this paper, mainly because the author’s familiarity with these topics is merely tangential. For example, recently, H. Bray has proved the Riemannian Penrose conjecture using the positive mass theorem (see Bray [11]).

As we mentioned earlier, the definition of (mass) energy stated is one of many, and it is the one that seems accepted in this area of study. It would be interesting to study the different formulations of mass and energy, and study analogous results (or counterexamples) if they exist. The definition given in this paper, known as the ADM mass, is said to be an appropriate definition. And in particular, it generalizes the Schwarchild mass, for example. For an interesting look at the notion of mass and energy in general relativity, one may wish to read R. Bartnik [12].

Appendix

We quickly review the notion of the second fundamental form associated to an immersed manifold \((M, g) \hookrightarrow (N, \gamma)\) of codimension one.

Let’s denote the Riemannian connection on \(N\) by \(\nabla\). Now, the splitting of the tangent space \(T_pN = T_pM \oplus T_pM^\perp\) allows us to define an induced connection \(\nabla\) on \(M\) by simply taking the tangential component of the connection \(\nabla\). That is, if \(X\) and \(Y\) are vector fields on \(M\), with local extensions \(X\) and \(Y\) on \(N\), then we may define \(\nabla\) by
\[
\nabla_X Y = (\nabla_X Y)^T,
\]
where the \(T\) denotes taking the component lying in \(T_pM\) with respect to the decomposition \(T_pN = T_pM \oplus T_pM^\perp\). Now we are ready to define the second fundamental form.

For the vector fields \(X\) and \(Y\) on \(M\) we define the bilinear symmetric map \(B(X, Y)\) by taking the normal component of the connection on \(N\):
\[
B(X, Y) = \nabla_X Y - (\nabla_X Y)^T = \nabla_X Y - \nabla_X Y
\]
Now, as the dimension of \(T_pM^\perp\) is one, let \(\eta\) its normal basis vector. The symmetric bilinear form \(H: T_pM \times T_pM \rightarrow \mathbb{R}\) given by
\[
H(X, Y) = \langle B(X, Y), \eta \rangle
\]
is called the second fundamental form associated to the imbedding \((M, g) \hookrightarrow (N, \gamma)\) at the point \(p \in M\).

We will be referring to the second fundamental form \(H\) by its coefficients \(H\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = h_{ij}\), to allow tensor manipulations in calculations.
References


