Fall 2006, MATC 25 Midterm Exam

Name ________________________________

Student ID ________________________________

Show all work clearly and in order, and circle your final answers. You have 110 minutes to finish this exam.

1 (40 points) True or False questions. Throughout the following questions we assume that the axioms in absolute geometry are valid. There are in total ten subquestions, each of which is worth four points.

(1) If ϕ is an isometry which fixes three distinct points on the plane, then ϕ is the identity map.
False. Any reflection fixes three disctint points on a line.

(2) The following statement is true in absolute geometry: an exterior angle of a triangle is equal in measure to the sum of the two non-adjacent interior angles of the triangle.
False. This is only true if the angles in a triangle sum to π.

(3) The existence of a quadrilateral whose angles sum to 2π implies the Euclid's fifth axiom.

True.

(4) Let P be a point outside a line l. Then there exists an unique line l_1 through p which is perpendicular to l.

True

(5) In hyperbolic geometry, given any triangle ABC, we have $|AB| + |BC| > |AC|$.

True. Line AC is the shortest path connecting two given point A and C.

(6) In hyperbolic geometry, if line l is perpendicular to two distinct lines l_1 and l_2, then l_1 and l_2 are ultra-parallel.

True.

(7) In hyperbolic geometry, if l_1 and l_2 are two lines which intersect at a point P, then there exists a line l which is parallel to both l_1 and l_2.

True. This is an equivalent way of stating that doubly asymptotic triangles exist in hyperbolic geometry.

(8) The image of y-axis under the inversion in the unit circle is y-axis itself.

True.

(9) Map $(x, y) \to (\frac{x+1}{(x+1)^2+y^2}, \frac{y}{(x+1)^2+y^2})$ is an isometry in the Poincaré upper plane model.

True. It is a composition of the horizontal translation with the inversion in the unit circle.

(10) Reflection in the y-axis is an isometry in the Poincaré upper plane model, but is not a direct isometry.

True.

2 (10 points) Let P be a point outside a line l. Prove that there exists a point Q on l such that the line l_1 connecting P and Q is perpendicular to l.

Proof: Let P' be the image of P under the reflection through the line l. Let l_1 be the line connecting P and P' and let Q be the intersection point of l' and l. We claim that Q is the desirable point we want. It suffices to check that l_1 is perpendicular to l. By construction, the image of l_1 under the reflection through the line l is l_1 itself. Also note that the image of l under the reflection through l is l itself. Thus the angles formed by l and l_1 have to be right angles.

3 (10 points) Prove that the map $(x, y) \to (1 - x, y)$ is an isometry in the Poincaré upper plane model.

Proof: The arc length is $ds = \sqrt{dx^2 + dy^2} y$.

Set $(u, v) = (1-x, y)$. Then we have $du = -dx$ and $dv = dy$. So $du^2 = dx^2$ and $dv^2 = dy^2$. Consequently $\frac{\sqrt{du^2+dv^2}}{v} = \frac{\sqrt{dx^2+dy^2}}{y}$.

4 (15 points) Let $A = 6i$ and $B = i$ be two points in the plane.

(a) (7 points) Suppose C is the point which lies one the hyperbolic line connecting A and B and which has the same hyperbolic distance from A and B. Find the coordinates of C.

Solution: Suppose $C = iy$. It follows from $|AC| = |CB|$ that $\ln \frac{6}{y} = \ln y$,i.e., $2 \ln y = \ln 6$. So $y = \sqrt{6}$.

(b) (8 points) Set $\gamma(z) = \frac{z}{z+1}$. First find the coordinates of $\gamma(A)$ and $\gamma(B)$ and then find the hyperbolic distance $|\gamma(A)\gamma(B)|$.

Solution:

$$\gamma(A) = \frac{6i}{6i+1} = \frac{36+6i}{37},$$
$$\gamma(B) = \frac{i}{1+i} = \frac{1+i}{2}.$$

Since γ is an element in $sl_2(R)$, it is an isometry in the Poincaré upper plane model. Consequently $|\gamma(A)\gamma(B)| = |AB| = \ln 6$.

5 (10 points) Let $A = i$ and $B = 1 + 2i$. Find the coordinate of the left end point of the semi-circle through A and B whose diameter lies on x-axis.

Solution: Suppose that the center of the circle is $(a, 0)$. Then we have $a^2+1 = (a-1)^2+4$. So $a = 2$ and the radius is $r = \sqrt{5}$. Thus the left end point is $(2-\sqrt{5}, 0)$.

6 (15 points)

(a) (7 points) Show that the image of the circle $C = \{(x, y) \in R^2, (x-1)^2 + y^2 = 1\}$ under the inversion in the unit circle is a vertical line.

Proof: Let C' be the image of C under the inversion. Then by the basic property of inversion in the unit circle C' is either a line or a circle. Note that the origin is a point on C and the image of it under the inversion is ∞. Therefore C' has to be a line. Since C is perpendicular to the x-axis and since the inversion preserves angles, C' is perpendicular to the x-axis.

(b) (8 points) Find the equation of the image of C under the inversion in the unit circle.

Solution: The right end point of C is $(2, 0)$. It is mapped to $(\frac{2}{2^2+0^2}, 0) = (\frac{1}{2}, 0)$ under the inversion in the unit circle. So the equation of C' is $x = \frac{1}{2}$.