SYMMETRIES IN GENERALIZED KÄHLER GEOMETRY

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ABSTRACT. We define the notion of a moment map and reduction in both generalized complex geometry and generalized Kähler geometry. As an application, we give very simple explicit constructions of bi-Hermitian structures on $\mathbb{CP}^N$, Hirzebruch surfaces, the blow up of $\mathbb{CP}^N$ at arbitrarily many points, and other toric varieties, as well as complex Grassmannians.

1. INTRODUCTION

Generalized complex structures were introduced by N. Hitchin [H02], and further developed by Gualtieri [Gua04]. It contains both symplectic and complex structures as extremal special cases, and provides a useful differential geometric context for understanding some recent development in string theory. An associated notion of generalized Kähler structure was introduced by Gualtieri [Gua04], who shows that this notion is essentially equivalent to that of a bi-Hermitian structure, which was first discovered by physicists (see [GHR84]) studying super-symmetric nonlinear σ-models.

The theory of bi-Hermitian geometry suffered from a lack of interesting examples. As stated in [AGG99] (see also [AGG05]), an important open problem in this field was to determine whether or not there exist bi-Hermitian structures on $\mathbb{CP}^2$, a minimal ruled surface admitting an effective anti-canonical divisor, or a complex surface obtained from them by blowing up points along an effective anti-canonical divisor. In a very recent paper [H05] Hitchin used the generalized Kähler geometric approach developed by Gualtieri [Gua04] to give an explicit construction of a bi-Hermitian structure on $\mathbb{CP}^2$ and also on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

For manifolds with symmetries, the related notions of moment maps and quotient are important in many geometries. It is an interesting question if there exist natural notions of moment maps and quotients in generalized complex and Kähler geometries. Some attempts have been made in this direction. In [Cr04], Crainic proposed a definition of moment maps in generalized complex geometry. But it appears that the condition to make his definition work is rather restrictive. In [H05] Hitchin also presented a quotient construction in generalized Kähler geometry which works for certain interesting special cases.

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In this paper, we define the notions of generalized moment map for a compact Lie group action on a generalized complex manifold. Using this definition we then define a generalized complex structure on the reduced space, which is natural up to transformation by an exact B-field, that is, the reduced space has a natural equivalence class of generalized complex structures in the sense specified in [H02]. Moreover, we show that the quotient structure has the same type as the original generalized complex structure. In the case that the generalized complex structure is derived from a symplectic structure, our definitions agree with the standard definitions of moment map and symplectic reduction. Compared with the definition of moment maps given in [Cr04], our approach works in greater generality.

We then consider the compact Lie group action on a generalized Kähler manifold; in this case, the generalized moment map is simply the generalized moment map for the first generalized complex structure. Finally, we define a natural generalized Kähler structure on the reduced space , and give formulas for the types of this structure. Again, in the case that the generalized Kähler structure is derived from a Kähler structure, this agrees with the usual Kähler reduction.

As an application, we give a very simple explicit construction of bi-Hermitian structures on $\mathbb{CP}^n$, Hirzebruch surfaces, the blow up of $\mathbb{CP}^N$ at arbitrarily many points, and other toric varieties, as well as complex Grassmannians. As shown in this paper, in practice our method gives a powerful machinery for producing bi-Hermitian structures on manifolds which can be produced as the symplectic quotient of $\mathbb{C}^N$.

Finally, we show that all our results carry over to the twisted case. In particular, we define twisted generalized complex reduction and twisted generalized Kähler reduction. We hope that this goes some way towards providing the framework which Kapustin and Li suggested would be useful [KL].

We would like to mention that in addition to the many concrete instances where we quote Gualtieri, our whole perspective on generalized geometries was heavily influenced by his excellent thesis on this subject.

Shortly before posting, we discovered that several other groups are independently working on related projects, including: Bursztyn, Cavalcanti, and Gualtieri [BCG05]; and Mathieu Stiénon and Xu Ping [SX05]. The latter paper [SX05] seems rather different in both perspective and techniques. In particular, instead of working with generalized moment maps, they consider quotients of arbitrary subsets; so our theorems do not appear in their paper (or vise-versa). They also do not construct bi-Hermitian structures. The results in [BCG05] also different from ours in several respects. First, whereas we define moment map and reduction for group actions on generalized complex manifolds, they define moment maps and reduction for "extended actions" on Courant algebroids and Dirac structures. In particular, their construction includes cases which we do not consider, such as complex quotients. On the other hand, the version of their paper which was contemporaneous with ours does not contain the case which we do consider here [1], and they do not have a formula for the type of the reduced structures. Additionally, they produce two bi-hermitian structures on $\mathbb{CP}^2$, but do not construct them on the other manifolds which we consider. Nevertheless, it is straightforward (though not entirely trivial) to check that the two notions of reduction are related. Namely, in the context of Proposition A.7, the map $\mathfrak{g} \oplus \mathfrak{g} \to C^\infty(TM \oplus T^*M)$ given by $(X, Y) \mapsto X_M + \alpha^X + df^Y$ defines an extended action in their sense, f is the moment map for this action, the reduced Courant algebroid is exact, and J induces a reduced generalized complex structure on the reduced algebroid which is isomorphic to the one defined in our paper. A similar remark applies to Proposition A.7.

Additionally, immediately after we posted, we learned that Shengda Hu also wrote a related paper [Hu05] which was partially inspired by an early version of this manuscript which we gave him in early June, and which includes a notion of twisted complex reduction which is very similar to ours in the untwisted case. More generally, this paper considers twisted complex structures in the framework of Hamiltonian symmetry. Our appendix was added after this appeared. Our main motivation is to demonstrate twisted generalized Kähler reduction, which is not in [Hu05]. However, it is also worth noting that even in the twisted generalized complex case our results are slightly different.

Finally, our work fits within a larger framework of literature on reduction. Most obviously, it a direct generalization of symplectic [MW74] and Kähler reduction [GS82]. Additionally, every generalized complex structure gives rise to a Poisson structure, and the Poisson structure associated to the reduced generalized complex structure is the reduced Poisson structure, as described in [MR86]. Finally, since a generalized complex structure is a Dirac structure of index zero, our results are certainly related to the body of work on the reduction of Dirac structures, such as [BL00], [BS01], [BR04],[MY05a], [MY05b].

The plan of this paper is as follows.

Section 2 shows that under reasonable assumptions the Courant bracket is preserved under restriction and quotient.

Section 3 defines generalized moment map for a compact Lie group acting on a generalized complex manifold, and constructs a generalized complex structure on the reduced space at every regular value.

Section 4 extends the results of Section 3 to the generalized Kähler case. We also discuss the connection between the hyper-Kähler quotient and the generalized Kähler quotient.

Section 5 presents the explicit constructions of bi-Hermitian structures.

[1] This has been added to recent versions.

2. THE COURANT BRACKET

Let V be a n dimensional vector space. There is a natural metric of type (n, n) on $V \oplus V^*$ given by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(Y)).$$

Given a subspace $F \subset V \oplus V^*$, let $F^\perp \subset V \oplus V^*$ denote the perpendicular with respect to this metric. In contrast, if $F \subset V$ (or V^*), let $F^0 \subset V^*$ (or V) denote the annihilator of F. Moreover, let $\pi : V_\mathbb{C} \oplus V_\mathbb{C}^* \to V_\mathbb{C}$ denote the natural projection.

Let M be a n dimensional manifold. There is a natural metric of type (n, n) on $TM \oplus T^*M$ given by

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\alpha(Y) + \beta(Y)),$$

which extends naturally to $T_\mathbb{C}M \oplus T_\mathbb{C}^*M = (TM \oplus T^*M) \otimes \mathbb{C}$.

Given a subbundle $F \subset TM \oplus T^*M$ (or $T_\mathbb{C}M \oplus T_\mathbb{C}^*M$), let $F^\perp \subset TM \oplus T^*M$ or $(T_\mathbb{C}M \oplus T_\mathbb{C}^*M)$ denote the perpendicular of F with respect to the above metric. Moreover, let $\pi : T_\mathbb{C}M \oplus T_\mathbb{C}^*M \to T_\mathbb{C}M$ denote the natural projection.

The **Courant bracket** on $T_\mathbb{C}M \oplus T_\mathbb{C}^*M$ is defined by

$$[X + \alpha, Y + \beta] = [X, Y] + L_X\beta - L_Y\alpha - \frac{1}{2}(d\iota_X\beta - d\iota_Y\alpha).$$

We will now examine how the Courant bracket behaves under restriction and quotient.

Lemma 2.1. *Let M be a manifold and let $\mathfrak{g}^*$ be the dual of a vector space $\mathfrak{g}$. Given a submersion $f \colon M \to \mathfrak{g}^*$, let $df \subset TM \oplus T^*M$ denote the subbundle spanned by the differentials df^ξ for $\xi \in \mathfrak{g}$. Then $df_\mathbb{C}^\perp$ is closed under the Courant bracket.*

Moreover, the restriction from M to $f^{-1}(0)$ induces a natural map from $df_\mathbb{C}^\perp \subset T_\mathbb{C}M \oplus T_\mathbb{C}^*M$ to $T_\mathbb{C}(f^{-1}(0)) \oplus T_\mathbb{C}^*(f^{-1}(0))$. If Γ is a sub-bundle of $T_\mathbb{C}M \oplus T_\mathbb{C}^*M$ which is closed under the Courant bracket, then the image of $\Gamma \cap df_\mathbb{C}^\perp$ under this map is also closed under the Courant bracket.

Proof. Let $X + \alpha$ and $Y + \beta$ be sections of $df_\mathbb{C}^\perp$. Given any $\xi \in \mathfrak{g}$, by assumption $\iota_X df^\xi = \iota_Y df^\xi = 0$. Hence by Cartan's Formula

$$\iota_{[X,Y]} df^\xi = L_X \iota_Y df^\xi - \iota_Y \iota_X ddf^\xi - \iota_Y d\iota_X df^\xi = 0.$$

Therefore, $[X + \alpha, Y + \beta] \in df_\mathbb{C}^\perp$. This proves the first claim.

Finally, if Γ is closed under the Courant bracket, then $\Gamma \cap df_\mathbb{C}^\perp$ is also closed. Since a straightforward check of the definition shows that map induced by restriction preserves the Courant bracket, the second claim is obvious. □

Lemma 2.2. *Let a compact Lie group G act freely on a manifold M, and let $\mathfrak{g}_M \subset T_\mathbb{C}M \oplus T_\mathbb{C}^*M$ denote the subbundle spanned by the fundamental vector fields ξ_M*

for ξ *is in the Lie algebra* $\mathfrak{g}$ *of* G. *Then the set of* G*-invariant sections of* $(\mathfrak{g}_M)^{\perp}_{\mathbb{C}}$ *is also closed under the Courant bracket.*

Moreover, the quotient map from M *to* M/G *induces a natural map from the set of* G*-invariant sections of* $(\mathfrak{g}_M)^{\perp}_{\mathbb{C}} \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ *to the section of* $T_{\mathbb{C}}(M/G) \oplus T^*_{\mathbb{C}}(M/G)$. *Let* Γ *be an* G*-invariant sub-bundle of* $T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ *which is closed under the Courant bracket. Then the image of* $\Gamma \cap (\mathfrak{g}_M)^{\perp}_{\mathbb{C}}$ *under this map is also closed under the Courant bracket.*

Proof. Let $X + \alpha$ and $Y + \beta$ be G-invariant sections of $(\mathfrak{g}_M)^{\perp}_{\mathbb{C}}$. Given any $\xi \in \mathfrak{g}$, by assumption $\iota_{\xi_M}\alpha = \iota_{\xi_M}\beta = 0$. Since $X + \alpha$ and $Y + \beta$ are G invariant, $[\xi_M, X] = 0$ and $L_{\xi_M}\iota_X\beta = 0$. Therefore

$$\iota_{\xi_M} L_X\beta = \iota_{[\xi_M, X]}\beta + L_X\iota_{\xi_M}\beta = 0 \quad \text{and} \quad \iota_{\xi_M} d\iota_X\beta = L_{\xi_M}\iota_X\beta + d\iota_X\iota_{\xi_M}\beta = 0.$$

Similarly, $\iota_{\xi_M} L_Y\alpha = \iota_{\xi_M} d\iota_Y\alpha = 0$. Hence, $[X + \alpha, Y + \beta] \in (\mathfrak{g}_M)^{\perp}_{\mathbb{C}}$.

Finally, if Γ is closed under the Courant bracket, then $\Gamma \cap (\mathfrak{g}_M)^{\perp}_{\mathbb{C}}$ is also closed. Since a straightforward check of the definition shows that the map induced by the quotient map preserves the Courant bracket, the second claim is obvious. □

3. Generalized complex structures

A **generalized complex structure** on a vector space V is an orthogonal linear map $\mathcal{J}\colon V \oplus V^* \to V \oplus V^*$ so that $\mathcal{J}^2 = -1$. Given a generalized complex structure $\mathcal{J}$, let $L \subset V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ be the $\sqrt{-1}$ eigenspace of $\mathcal{J}$. Then L is maximal isotropic and $L \cap \overline{L} = \{0\}$. Conversely, given a maximal isotropic $L \subset V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ so that $L \cap \overline{L} = \{0\}$, there exists a unique generalized complex structure on V whose $\sqrt{-1}$ eigenspace is L.

Let $\mathcal{J}$ be a generalized complex structure on a vector space V and let $W = V \oplus V^*$. If $P \subset W$ is a $\mathcal{J}$-invariant subspace, then since $\mathcal{J}$ is orthogonal there is a restriction map $\overline{\mathcal{J}}\colon P^{\perp} \to P^{\perp}$. If P is also isotropic, let $\widetilde{W} = P^{\perp}/P$; there is a quotient map $\widetilde{\mathcal{J}}\colon \widetilde{W} \to \widetilde{W}$. Clearly, $\overline{\mathcal{J}}^2 = -1$, $\widetilde{\mathcal{J}}^2 = -1$ and both maps are orthogonal. Also, if L is the $\sqrt{-1}$ eigenbundle of J, then $L \cap P^{\perp}_{\mathbb{C}}$ is the $\sqrt{-1}$ eigenbundle of $\overline{\mathcal{J}}$, and the image of $L \cap P^{\perp}_{\mathbb{C}}$ in $\widetilde{W}_{\mathbb{C}}$ is the $\sqrt{-1}$ eigenbundle of $\widetilde{\mathcal{J}}$. Finally, if $P = (P \cap V) \oplus (P \cap V^*)$, let $\widetilde{V}$ be the quotient of $(P \cap V^*)^0 \subset V$ by $(P \cap V)$; the spaces $\widetilde{W}$ and $\widetilde{V} \oplus \widetilde{V}^*$ are naturally isomorphic. Hence $\mathcal{J}$ naturally induces a generalized complex structure $\widetilde{\mathcal{J}}$ on $\widetilde{V}$.

The **type** of $\mathcal{J}$ is the codimension of $\pi(L)$ in $V_{\mathbb{C}}$, where L is the $\sqrt{-1}$ eigenspace of $\mathcal{J}$. (Recall that $\pi\colon V_{\mathbb{C}} \oplus V^*_{\mathbb{C}} \to V_{\mathbb{C}}$ is the natural projection.) The following lemma will help us compute types.

Lemma 3.1. *Let* $\mathcal{J}$ *be a generalized complex structure on a vector space* V, *and let* $L \subset V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ *be its* $\sqrt{-1}$ *eigenspace. If a subspace* $R \subset V_{\mathbb{C}} \oplus V^*_{\mathbb{C}}$ *satisfies* $\mathcal{J}(R) \cap R = \{0\}$, *then*

$$\dim(\pi(L \cap R^{\perp} \cap \mathcal{J}(R)^{\perp})) = \dim(\pi(L + R)) - \dim(R).$$

Proof. Since L is the $\sqrt{-1}$ eigenspace of $\mathcal{J}$,

$$L \cap R^{\perp} \cap \mathcal{J}(R)^{\perp} = L \cap R^{\perp}.$$

Since $L \cap R^{\perp} \cap \mathcal{J}(R)^{\perp}$ is the $\sqrt{-1}$ eigenspace of the restriction of $\mathcal{J}$ to $R^{\perp} \cap \mathcal{J}(R)^{\perp}$ and $\mathcal{J}(R) \cap R = \{0\}$,

$$\dim(L \cap R^{\perp} \cap \mathcal{J}(R)^{\perp}) = \dim V - \dim R.$$

Since $V^*_{\mathbb{C}}$ is the kernel of π,

$$\dim(\pi(L \cap R^{\perp})) = \dim(L \cap R^{\perp}) - \dim(L \cap R^{\perp} \cap V^*_{\mathbb{C}}).$$

Finally, since L is maximal isotropic, $L = L^{\perp}$, and so

$$L \cap R^{\perp} \cap V^*_{\mathbb{C}} = (L + R)^{\perp} \cap V^*_{\mathbb{C}} = \pi(L + R)^0.$$

□

Lemma 3.2. *Let $\mathcal{J}$ be a generalized complex structure on a vector space V. Consider a subspace $Q \subset V$ so that $\mathcal{J}(Q) \subset V^*$ and so that $P = Q \oplus \mathcal{J}(Q) \subset V \oplus V^*$ is isotropic. Let $\widetilde{\mathcal{J}}$ be the natural generalized complex structure on $\widetilde{V} = \mathcal{J}(Q)^0/Q$. Then*

$$\operatorname{type}(\widetilde{\mathcal{J}}) = \operatorname{type}(\mathcal{J}).$$

Proof. Let L and $\widetilde{L}$ be the $\sqrt{-1}$ eigenspaces of $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, respectively; let $\pi\colon V_{\mathbb{C}} \oplus V^*_{\mathbb{C}} \to V_{\mathbb{C}}$ and $\widetilde{\pi}\colon \widetilde{V}_{\mathbb{C}} \oplus \widetilde{V}^*_{\mathbb{C}} \to \widetilde{V}_{\mathbb{C}}$ be the natural projections.

Since $Q \subset V$ and $\mathcal{J}(Q) \subset V^*$, it is immediately clear that $Q_{\mathbb{C}} \subset \pi(L)$ and $Q \cap \mathcal{J}(Q) = \{0\}$. Therefore, by Lemma 3.1 $\dim(\pi(L \cap Q_{\mathbb{C}}^{\perp} \cap \mathcal{J}(Q_{\mathbb{C}})^{\perp})) = \dim(\pi(L)) - \dim(Q)$. Moreover, $\widetilde{\pi}(\widetilde{L})$ is the projection of $\pi(L \cap Q_{\mathbb{C}}^{\perp} \cap \mathcal{J}(Q_{\mathbb{C}})^{\perp}) \subset \mathcal{J}(Q)^0_{\mathbb{C}}$ to $\widetilde{V}_{\mathbb{C}} = \mathcal{J}(Q)^0_{\mathbb{C}}/Q_{\mathbb{C}}$, so $\dim(\widetilde{\pi}(\widetilde{L})) = \dim(\pi(L \cap Q_{\mathbb{C}}^{\perp} \cap \mathcal{J}(Q_{\mathbb{C}})^{\perp})) - \dim(Q)$. Finally, $\dim(\widetilde{V}) = \dim(V) - 2\dim(Q)$. □

A **generalized almost complex structure** on a manifold M is an orthogonal bundle map $\mathcal{J}\colon TM \oplus T^*M \to TM \oplus T^*M$ so that $\mathcal{J}^2 = -1$. Moreover, $\mathcal{J}$ is a **generalized complex structure** if the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$, $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$, is closed under the Courant bracket. The **type** of $\mathcal{J}$ at $m \in M$ is the type of the restricted generalized complex structure on T_mM.

We now introduce several standard examples, as described in [Gua04].

Example 3.3. ([H02])

a) Let (M, ω) be a symplectic manifold. Then

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix}$$

is a generalized complex structure of type 0; the $\sqrt{-1}$ eigenbundle of $\mathcal{J}_\omega$ is $L_\omega = \{X + \sqrt{-1}\,\iota_X\omega \mid X \in T_{\mathbb{C}}M\}$.

b) Let (M, J) be a $2n$ dimensional complex manifold. Then

$$\mathcal{J}_J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}$$

is a generalized complex structure of type n; the $\sqrt{-1}$ eigenbundle of $\mathcal{J}_J$ is $L_J = T_{0,1} \oplus T^*_{1,0}$, where $T_{1,0}$ is the $\sqrt{-1}$ eigenbundle of J.

c) Let $(M_1, \mathcal{J}_1)$ and $(M_2, \mathcal{J}_2)$ be generalized complex manifolds. Then

$$\mathcal{J}_1 \times \mathcal{J}_2 = \begin{pmatrix} \mathcal{J}_1 & 0 \\ 0 & \mathcal{J}_2 \end{pmatrix}.$$

is a generalized complex structure on $M_1 \times M_2$, and

$$\text{type}(\mathcal{J}_1 \times \mathcal{J}_2)_{(m_1,m_2)} = \text{type}(\mathcal{J}_1)_{m_1} + \text{type}(\mathcal{J}_2)_{m_2}.$$

d) Given a two-form B on a manifold M, consider the orthogonal bundle map $TM \oplus T^*M \to TM \oplus T^*M$ defined by

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix},$$

where B is regarded as a skew-symmetric map from TM to T^*M. If $\mathcal{J}$ is a generalized almost complex structure on M, then $\mathcal{J}' = e^B \mathcal{J} e^{-B}$ is another generalized almost complex structure on M, called the **B-transform** of $\mathcal{J}$. We say that the B-transform is **closed** (or **exact**) if B is. If L is the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$, then $e^B(L)$ is the $\sqrt{-1}$-eigenbundle of $\mathcal{J}'$, so $\mathcal{J}$ and $\mathcal{J}'$ have the same type. If B is closed, then e^B preserves the Courant bracket (see [Gua04]). Thus, if $\mathcal{J}$ is a generalized complex structure, then so is $\mathcal{J}'$.

Definition 3.4. *Let a compact Lie group G with Lie algebra $\mathfrak{g}$ act on a manifold M, preserving a generalized complex structure $\mathcal{J}$. Let $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ denote the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$. A* **generalized moment map** *is a smooth function $\mu \colon M \to \mathfrak{g}_{\mathbb{C}}$ so that*

- *$\xi_M - \sqrt{-1}\, d\mu^\xi$ lies in L for all $\xi \in \mathfrak{g}$, where ξ_M denotes the induced vector field on M.*
- *μ is equivariant.*

The generalized moment map μ is **real** *if $\mu = \overline{\mu}$. The action is* **Hamiltonian** *if a generalized moment moment map exists.*

If $\mu = f + \sqrt{-1}\, g$, where f and g are real, the first condition is equivalent to $\mathcal{J}(df^\xi) = -\xi_M - dg^\xi$. For S^1 actions, this is similar to the notion of Hamiltonian defined in [Gua04].

Gualtieri observed (see also [AB04]) that for any generalized complex manifold $(M, \mathcal{J})$, the bivector Π defined by the upper right quadrant of $\mathcal{J} \colon TM \oplus T^*M \to TM \oplus T^*M$ is a real Poisson bivector; this gives rise to a Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$. A simple calculation shows that if a compact Lie group G acts on $(M, \mathcal{J})$ with generalized moment map $\mu =$

$f + \sqrt{-1}\,h$, where f and h are real, then f is an (equivariant) moment map for the action on the Poisson manifold (M, Π).

If $\mu : M \to \mathfrak{g}^*_{\mathbb{C}}$ is a generalized moment map, and $\phi : M \to \mathfrak{g}^*_{\mathbb{C}}$ is an equivariant map, then clearly $\mu + \phi$ is a generalized moment map exactly if $d\phi^\xi \in L$ for all $\xi \in \mathfrak{g}$. Since L is maximal isotropic, this occurs exactly if $d\phi^\xi$ vanishes on $\pi(L)$. For example, if $\mathcal{J}$ has type 0 then the generalized moment map is unique, up to a constant.

It will be convenient to have the following definition.

Definition 3.5. *Let a compact Lie group* G *act on a manifold* M. *The* **Cartan model** *for the equivariant cohomology of* M *is defined as follows: The degree* n *co-chains are*

$$\Omega^n_G(M) = \bigoplus_i (\Omega^{n-2i}(M) \otimes S^i(\mathfrak{g}^*))^G,$$

where S^i *denotes polynomials of degree* i. *The differential* $d_G : \Omega^n_G \to \Omega^{n+1}_G$ *is defined by*

$$d_G(\alpha \otimes p)(\xi) = (d\alpha - \iota_{\xi_M}\alpha)p(\xi) \qquad \textit{for all } \xi \in \mathfrak{g},$$

where we think of $\Omega^*_G(M)$ *as the space of equivariant polynomial mappings from* $\mathfrak{g}$ *to* $\Omega^*(M)$. *(If* G *acts on a vector space* A, *let* A^G *denote the invariant subspace.) The* **equivariant cohomology** *of* M *is* $H^*_G(M) = H^*(\Omega^*_G, d_G)$.

Remark 3.6. Let a compact Lie group G act on a manifold M so that it acts freely on a submanifold $L \subset M$. There is a natural map, called the **Kirwan map**

$$\kappa : H^*_G(M) \to H^*(L/G)$$

which is the composition of the restriction map from $H^*_G(M)$ to $H^*_G(L)$ with the natural isomorphism from $H^*_G(L)$ to $H^*(L/G)$.

A form $B \in \Omega^n(M)$ is **basic** if it is invariant and $\iota_{\xi_M} B = 0$ for all $\xi \in \mathfrak{g}$. Then B **descends** to $\widetilde{B} \in \Omega^n(L/G)$, that is, the pull-back of $\widetilde{B}$ to L is the restriction of B. If $B \in \Omega^n(M)^G \subset \Omega^n_G(M)$ is **equivariantly closed**, that is, $d_G B = 0$, then B is closed and basic and $\kappa[B] = [\widetilde{B}]$. More generally, if $\eta \in \Omega^n_G(M)$ is equivariantly closed, there exists $\Gamma \in \Omega^{n-1}_G(L)$ so that $\eta|_L + d_G\Gamma \in \Omega^n(L)^G \subset \Omega^n_G(L)$. Since $\eta + d_G\Gamma$ is equivariantly closed, it descends to $\widetilde{\eta} \in \Omega^n(L/G)$ and $\kappa[\eta] = [\widetilde{\eta}]$.

We now compute generalized moment maps for our basic examples.

Example 3.7.

a) Let G act on a symplectic manifold (M, ω) with moment map $\Phi \colon M \to \mathfrak{g}^*$, that is, Φ is equivariant and $\iota_M\omega = d\Phi^\xi$ for all $\xi \in \mathfrak{g}$. Then G also preserves the generalized complex structure $\mathcal{J}_\omega$, and Φ is a real generalized moment map for this action.

b) Let G act on a complex manifold (M, J), preserving J. Then G also preserves the generalized complex structure $\mathcal{J}_J$. However, non-trivial actions are never Hamiltonian because $\pi(L_J)$ contains no non-trivial real vectors.

c) Let G act on generalized complex manifolds $(M_1, \mathcal{J}_1)$ and $(M_2, \mathcal{J}_2)$ with generalized moment maps μ_1 and μ_2. Then the diagonal action of G on the product manifold $M_1 \times M_2$ preserves the generalized complex structure $\mathcal{J}_1 \times \mathcal{J}_2$ with generalized moment map $\mu = \mu_1 + \mu_2 \colon M_1 \times M_2 \to \mathfrak{g}_{\mathbb{C}}$.

d) Let G act on a generalized complex manifold $(M, \mathcal{J})$ with generalized moment map $f + \sqrt{-1}\, h \colon M \to \mathfrak{g}^*_{\mathbb{C}}$, where f and h are real. Given a closed form $B \in \Omega^2(M)^G$, G also preserves the B-transform $\mathcal{J}'$ of $\mathcal{J}$. If $\Phi \colon M \to \mathfrak{g}^*$ is an equivariant map, then $f + \sqrt{-1}\,(h + \Phi)$ is a generalized moment map for $\mathcal{J}'$ exactly if $B + \Phi \in \Omega^2_G(M)$ is closed, that is, $\iota_{\xi_M} B = d\Phi^{\xi}$ for all $\xi \in \mathfrak{g}$. We can think of $(M, \mathcal{J}', f + \sqrt{-1}\,(h + \Phi))$ as the $B + \Phi$- transform of $(M, \mathcal{J}, f + \sqrt{-1}\, h)$.

In particular, if G acts freely on M then we can always perform an exact B-transform so that the generalized moment map is real. Let $\theta \in \Omega^1(M, \mathfrak{g})$ be a connection, and define $\Gamma = (\theta, h) \in \Omega^1(M)^G$ using the natural pairing of $\mathfrak{g}$ with $\mathfrak{g}^*$. Since θ is a connection, $\iota_{\xi_M} \Gamma = h^{\xi}$ for all $\xi \in \mathfrak{g}$. So $d_G \Gamma = d\Gamma - h$ is equivariantly closed and thus f is a generalized moment map for the $d\Gamma$-transform of $\mathcal{J}$.

Lemma 3.8. *Let a compact Lie group G act on a generalized complex manifold $(M, \mathcal{J})$ with a real generalized moment map $f \colon M \to \mathfrak{g}^*$. Let $\mathcal{O}_a$ be a co-adjoint orbit through $a \in \mathfrak{g}^*$ so that G acts freely on $f^{-1}(\mathcal{O}_a)$. Then the generalized complex quotient M_a inherits a natural generalized complex structure $\widetilde{\mathcal{J}}$.*

Moreover, for all $m \in f^{-1}(\mathcal{O}_a)$,

$$\operatorname{type}(\widetilde{\mathcal{J}})_{[m]} = \operatorname{type}(\mathcal{J})_m.$$

Proof. First, assume that $a = 0$.

By restricting to a neighborhood of $f^{-1}(0)$, we may assume that G acts freely, and that hence f is a submersion. By the definition of generalized moment map, $\mathcal{J}(\xi_M) = df^{\xi}$ for all $\xi \in \mathfrak{g}$, so $\mathcal{J}(\mathfrak{g}_M) = df$. Therefore, $\mathfrak{g}_M \oplus df$ is a $\mathcal{J}$-invariant subbundle of $TM \oplus T^*M$. Since G acts on $f^{-1}(0)$, $\mathfrak{g}_M \oplus df$ is also isotropic when restricted to $f^{-1}(0)$. As in the discussion preceding Lemma 3.2, $\mathcal{J}$ naturally induces a G equivariant orthogonal map with square -1 on the G-invariant vector bundle

$$(\mathfrak{g}_M \oplus df)|^{\perp}_{f^{-1}(0)} / (\mathfrak{g}_M \oplus df)|_{f^{-1}(0)}.$$

Let $\widetilde{\mathcal{J}} \colon TM_0 \oplus T^*M_0 \to TM_0 \oplus T^*M_0$ be the induced generalized almost complex structure on M_0.

Let $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ and $\widetilde{L} \subset T_{\mathbb{C}}M_0 \oplus T^*_{\mathbb{C}}M_0$ be the $\sqrt{-1}$ eigenbundles of $\mathcal{J}$ and $\widetilde{\mathcal{J}}$, respectively. By the definition of generalized complex structure, L is closed under the Courant bracket. By Lemma 2.1, the image of $L \cap df^{\perp}$ in $T_{\mathbb{C}}(f^{-1}(0)) \oplus T^*_{\mathbb{C}}(f^{-1}(0))$ is also closed under the Courant bracket. Since

$L \cap df^\perp = L \cap (\mathfrak{g}_M \oplus df)^\perp$, by Lemma 2.2 $\widetilde{L}$, which is its image in $T_{\mathbb{C}}M_0 \oplus T^*_{\mathbb{C}}M_0$, is also closed.

The last statement is a direct consequence of Lemma 3.2.

This proves the case $a = 0$.

For $a \neq 0$, let ω be the Kirillov-Kostant symplectic form on the co-adjoint orbit $\mathcal{O}_{-a}$ and let $\mathcal{J}_\omega$ be the induced generalized complex structure. Then $(\mathcal{O}_{-a}, \mathcal{J}_\omega)$ is a generalized complex manifold of type 0, and inclusion is a generalized moment map for the co-adjoint G action. Hence, $(M \times \mathcal{O}_{-a}, \mathcal{J} \times \mathcal{J}_\omega)$ is a generalized complex manifold, $\text{type}(\mathcal{J} \times \mathcal{J}_\omega)_{(m,b)} = \text{type}(\mathcal{J})_m$, and $\mu_a(x, \nu) = \mu(x) - \nu$ is a generalized moment map for the diagonal action of G on $M \times \mathcal{O}_{-a}$. Since it is easy to see that M_a can be identified with $\mu_a^{-1}(0)/G$, the result follows from the case $a = 0$. □

We now find the generalized complex quotients for the examples in 3.7.

Example 3.9.

a) The generalized complex quotient of the Hamiltonian generalized complex manifold associated to a Hamiltonian symplectic manifold is the generalized complex manifold associated to the symplectic quotient.
b) Since there is no generalized moment map in the complex case, there is no generalized complex quotient.
c) As in the symplectic case, the generalized complex quotient for the diagonal action on the product of two generalized complex manifolds is not the product of the quotients.
d) Let G act on a generalized complex manifold $(M, \mathcal{J})$ with a real generalized moment map f. Let $B \in \Omega^2(M)$ be closed and basic Then for any regular value $a \in \mathfrak{g}^*$, B descends to $\widetilde{B} \in \Omega^2(M_a)$ The generalized complex quotient of the B-transform of $(M, \mathcal{J}, f)$ is the $\widetilde{B}$-transform of the generalized complex quotient of $(M, \mathcal{J}, f)$. Additionally, if $B = d\gamma$, where $\gamma \in \Omega^1(M)$ is basic and descends to $\widetilde{\gamma} \in \Omega^1(M_a)$, then $\widetilde{B} = d\widetilde{\gamma}$.

Our first main result is now very easy to prove.

Proposition 3.10. *Let a compact Lie group* G *act on a generalized complex manifold* $(M, \mathcal{J})$ *with generalized moment map* $\mu = f + \sqrt{-1}h\colon M \to \mathfrak{g}^*_{\mathbb{C}}$, *where* f *and* h *are real. Let* $\mathcal{O}_a$ *be a co-adjoint orbit through* $a \in \mathfrak{g}^*$ *so that* G *acts freely on* $f^{-1}(\mathcal{O}_a)$. *Then the generalized complex quotient* M_a *inherits a generalized complex structure* $\widetilde{\mathcal{J}}$, *which is natural up to an exact* B*-transform. Moreover, for all* $m \in f^{-1}(\mathcal{O}_a)$,

$$\text{type}(\widetilde{\mathcal{J}})_{[m]} = \text{type}(\mathcal{J})_m.$$

Proof. By restricting to a neighborhood of $f^{-1}(\mathcal{O}_a)$, we may assume that G acts freely. Choose a connection $\theta \in \Omega^1(M, \mathfrak{g})$. As in part (d) of Example 3.7, the $d(h, \theta)$-transform of $\mathcal{J}$ is an invariant generalized complex structure with real moment map f, and hence descends to a natural generalized complex structure $\widetilde{\mathcal{J}}$ on M_a by Lemma 3.8.

If $\theta' \in \Omega^1(M, \mathfrak{g})$ is a different connection, then $\gamma = (h, \theta' - \theta)$ is basic and hence descends to $\widetilde{\gamma}$. Hence, the resulting generalized complex structure on M_a is the $d\widetilde{\gamma}$ transform of $\widetilde{\mathcal{J}}$. $\square$

Example 3.11. Let G act on a generalized complex manifold $(M, \mathcal{J})$ with generalized moment map $\mu : M \to \mathfrak{g}^*_{\mathbb{C}}$ and generalized complex quotient $(M_a, \widetilde{\mathcal{J}})$. Fix a connection one form θ on M.

Given an equivariantly closed form $B + \Phi \in \Omega^2_G(M)$, $B + \Phi + d_G(\theta, \Phi) = B + d(\theta, \Phi) \in \Omega^2(M)^G$ is equivariantly closed and hence descends to a closed form $\widetilde{B} \in \Omega^2(M_a)$. Note that $[\widetilde{B}]$ is the image of $[B + \Phi]$ under the Kirwan map. If $\mathcal{J}'$ is the B-transform of $\mathcal{J}$, then the generalized complex quotient of $(M, \mathcal{J}', \mu + \sqrt{-1}\,\Phi)$ is the $\widetilde{B}$ transform of $\widetilde{\mathcal{J}}$. That is, an equivariantly closed transform descends to a closed transform in the cohomology class of its image under the Kirwan map.

If there exists $\gamma \in \Omega^1(M)^G = \Omega^1_G(M)$ so that $B + \Phi = d_G\gamma$, then $\gamma - (\Phi, \theta)$ is basic and hence descends to a form $\widetilde{\gamma} \in \Omega^1(M_a)$ so that $d\widetilde{\gamma} = \widetilde{B}$, that is, equivariantly exact transforms descend to exact transforms. In contrast, if $B = 0$ and Φ is a non-zero constant $c \in \mathfrak{g}^*$, then $B = -(c, d\theta)$, and $\widetilde{B}$ is generally not exact.

4. Generalized Kähler structure

A **generalized Kähler structure** on a vector space V consists of an ordered pair $(\mathcal{J}_1, \mathcal{J}_2)$ of commuting generalized complex structures on V so that $K = -\mathcal{J}_1\mathcal{J}_2 \colon V \oplus V^* \to V \oplus V^*$ is a **positive definite metric**, by which we mean that $K^2 = 1$, K is orthogonal, and $\langle K(w), w \rangle > 0$ for all non-zero $w \in V \oplus V^*$. Note that the first two conditions are automatically satisfied.

We will need the following lemma:

Lemma 4.1. *Let V be a vector space and let $W = V \oplus V^*$. Let $K \colon W \to W$ be a positive definite metric. Given an isotropic subspace P, define*

$$\widehat{W} = P^\perp \cap K(P)^\perp \subset W \qquad \text{and} \qquad \widetilde{W} = P^\perp / P.$$

The natural projection induces an isomorphism

$$\widehat{W} \hookrightarrow\!\!\!\!\!\rightarrow \widetilde{W}. \tag{4.1}$$

Proof. For all non-zero $w \in P$, $\langle w, K(w) \rangle > 0$. Therefore, $K(P)^\perp \cap P = P^\perp \cap K(P) = \{0\}$. Since $\dim(\widehat{W}) = \dim W - 2 \dim P$. It also implies that the projection above is an injection; the result now follows by a dimension count. $\square$

Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on a vector space V, and let $K = -\mathcal{J}_1\mathcal{J}_2$. If $P \subseteq W = V \oplus V^*$ is any $\mathcal{J}_1$ invariant subspace, then since $\mathcal{J}_1 K = K \mathcal{J}_1$, $P \oplus K(P)$ is $\mathcal{J}_1$ and $\mathcal{J}_2$ invariant and so we can define the restrictions $\widehat{\mathcal{J}}_i$ of $\mathcal{J}_i$ to $\widehat{W} = W \cap P^\perp \cap K(P)^\perp$. Clearly, $\widehat{\mathcal{J}}_1$ and $\widehat{\mathcal{J}}_2$ are orthogonal and commute, $\widehat{\mathcal{J}}_1^2 = \widehat{\mathcal{J}}_2^2 = -1$, and $\widehat{K} = -\widehat{\mathcal{J}}_1\widehat{\mathcal{J}}_2$ is a positive definite metric on $\widehat{W}$. If P is also isotropic, then under the isomorphism (4.1) the $\widehat{\mathcal{J}}_i$ induce maps $\widetilde{\mathcal{J}}_i \colon \widetilde{W} \to \widetilde{W}$ satisfying the analogous conditions. Moreover, if L_i is the $\sqrt{-1}$ eigenspace of $\mathcal{J}_i$, then $L_i \cap \widehat{W}$ is the $\sqrt{-1}$ eigenspace of $\widehat{\mathcal{J}}_i$, and the $\sqrt{-1}$ eigenspace of $\widetilde{J}_i$ is its image under the isomorphism (4.1). As in the previous section, if $P = (P \cap V) \oplus (P \cap V^*)$ then $(\widetilde{\mathcal{J}}_1, \widetilde{\mathcal{J}}_2)$ is a natural generalized Kähler structure on $\widetilde{V} = (P \cap V^*)^0/(P \cap V)$. It is easy to check that $\widetilde{\mathcal{J}}_1$ is the natural complex structure on $\widetilde{V}$ defined in the beginning of the previous section.

Lemma 4.2. *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler structure on a vector space V. Consider $Q \subseteq V$ so that $\mathcal{J}_1(Q) \subseteq V^*$ and $P = Q \oplus \mathcal{J}_1(Q) \subset V \oplus V^*$ is isotropic. Let $(\widetilde{\mathcal{J}}_1, \widetilde{\mathcal{J}}_2)$ be the natural Kähler structure on $\widetilde{V} = \mathcal{J}_1(Q)^0/Q$. Then*

$$\operatorname{type}(\widetilde{\mathcal{J}}_1) = \operatorname{type}(\mathcal{J}_1) \qquad \textit{and}$$

$$\operatorname{type}(\widetilde{\mathcal{J}}_2) = \operatorname{type}(\mathcal{J}_2) - \dim(Q) + 2\dim(Q_{\mathbb{C}} \cap \pi(L_2)).$$

Proof. The first claim was proved in Lemma 3.2.

We now turn to the second claim. Let L_2 and $\widetilde{L}_2$ denote the $\sqrt{-1}$ eigenspaces of $\mathcal{J}_2$ and $\widetilde{\mathcal{J}}_2$ respectively; let $\pi \colon V_{\mathbb{C}} \oplus V_{\mathbb{C}}^* \to V_{\mathbb{C}}$ and $\widetilde{\pi} \colon \widetilde{V}_{\mathbb{C}} \oplus \widetilde{V}_{\mathbb{C}}^* \to \widetilde{V}_{\mathbb{C}}$ be the natural projections.

Since P is $\mathcal{J}_1$ invariant, $K(P) = \mathcal{J}_2(P)$, so $\mathcal{J}_2(P) \cap P = \{0\}$. Moreover, by assumption, $\pi(P_{\mathbb{C}}) = \pi(Q_{\mathbb{C}})$ and $\dim(P) = 2\dim(Q)$. Therefore, by Lemma 3.1, $\dim(\pi(L_2 \cap P_{\mathbb{C}}^\perp \cap \mathcal{J}_2(P_{\mathbb{C}})^\perp)) = \dim(\pi(L_2 + P_{\mathbb{C}})) - \dim(P) = \dim(\pi(L_2) + Q_{\mathbb{C}}) - 2\dim(Q) = \dim(\pi(L_2)) - \dim(Q) - \dim(Q_{\mathbb{C}} \cap \pi(L_2))$. Moreover, $\widetilde{\pi}(\widetilde{L}_2)$ is the projection of $\pi(L_2 \cap P_{\mathbb{C}}^\perp \cap \mathcal{J}_2(P_{\mathbb{C}})^\perp) \subset \mathcal{J}_1(Q_{\mathbb{C}})^0$ to $\widetilde{V}_{\mathbb{C}} = \mathcal{J}_1(Q_{\mathbb{C}})^0/Q_{\mathbb{C}}$, which reduces the dimension by a further $\dim(\pi(L_2) \cap Q_{\mathbb{C}})$. Finally, $\dim(\widetilde{V}) = \dim(V) - 2\dim(Q)$.

□

A **generalized Kähler structure** on a manifold M is a pair of commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on M so that $K = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $TM \oplus T^*M$. Note by definition the restriction of K to TM induces a canonical Riemannian metric k on the manifold M.

Our basic examples are taken from Gualtieri.

Example 4.3. [Gua04]

a) Let (ω, J) be a **genuine Kähler structure** on a manifold M, that is, a symplectic structure ω and a complex structure J which are **compatible**, which means that $g = -\omega J$ is a Riemannian metric. By Example 3.3, ω and J induce generalized complex structures $\mathcal{J}_\omega$ and $\mathcal{J}_J$, respectively. Moreover, it is easy to see that $\mathcal{J}_J$ and $\mathcal{J}_\omega$

commute, and that

$$-\mathcal{J}_\omega \mathcal{J}_J = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{4.2}$$

is a positive definite metric on $TM \oplus T^*M$. Hence $(\mathcal{J}_\omega, \mathcal{J}_J)$ is a generalized Kähler structure on M.

b) Let $(M, \mathcal{J}_{M,1}, \mathcal{J}_{M,2})$ and $(N, \mathcal{J}_{N,1}, \mathcal{J}_{N,2})$ be generalized Kähler manifolds, and define

$$\mathcal{J}_1 = \begin{pmatrix} \mathcal{J}_{M,1} & 0 \\ 0 & \mathcal{J}_{N,1} \end{pmatrix} \quad \text{and} \quad \mathcal{J}_2 = \begin{pmatrix} \mathcal{J}_{M,2} & 0 \\ 0 & \mathcal{J}_{N,2} \end{pmatrix}.$$

Then $(M \times N, \mathcal{J}_1, \mathcal{J}_2)$ is a generalized Kähler manifold.

Definition 4.4. *Let the compact Lie group G with Lie algebra $\mathfrak{g}$ act on a manifold M. A* **generalized moment map** *for an invariant generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ is a generalized moment map for the generalized complex structure $\mathcal{J}_1$. (See Definition 3.4.)*

As before, let a compact Lie group G act on a generalized Kähler manifold with generalized moment map $\mu = f + \sqrt{-1}\,h$, where f and h are real. Let $\mathcal{O}_a$ be the co-adjoint orbit through $a \in \mathfrak{g}^*$. If G acts freely on $f^{-1}(\mathcal{O}_a)$ then the **generalized Kähler quotient**

$$M_a = f^{-1}(\mathcal{O}_a)/G$$

is a manifold.

Example 4.5.

a) If a compact Lie group G acts on a Kähler manifold (M, J, ω) with moment map Φ, then Φ is the generalized moment map for the G action on $(M, \mathcal{J}_J, \mathcal{J}_\omega)$.

b) If a compact Lie group G acts on two generalized Kähler manifold $(M, \mathcal{J}_1^M, \mathcal{J}_2^M)$ and $(N, \mathcal{J}_1^N, \mathcal{J}_2^N)$ with moment maps μ^N and μ^N, then $\mu^M + \mu^N$ is a generalized moment map for the diagonal G action on $(M \times N, \mathcal{J}_1^M \times \mathcal{J}_1^N, \mathcal{J}_2^M \times \mathcal{J}_2^N)$.

We can now state our second main proposition:

Proposition 4.6. *Let a compact connected Lie group G act on a generalized Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ with generalized moment map $\mu = f + \sqrt{-1}\,h\colon M \to \mathfrak{g}^*_{\mathbb{C}}$. Let $\mathcal{O}_a$ be a co-adjoint orbit through $a \in \mathfrak{g}^*$ so that G acts freely on $f^{-1}(\mathcal{O}_a)$. Then the generalized Kähler quotient M_a naturally inherits a generalized Kähler structure $(\widetilde{\mathcal{J}}_1, \widetilde{\mathcal{J}}_2)$.*

Moreover, let $\mathfrak{h}$ be the Lie algebra of the stabilizer H of a, and let L_2 be the $\sqrt{-1}$ eigenbundle of $\mathcal{J}_2$. Then for all $m \in M$,

$$\text{type}(\widetilde{\mathcal{J}}_1)_{[m]} = \text{type}(\mathcal{J}_1)_m, \qquad \textit{and}$$

$$\text{type}(\widetilde{\mathcal{J}}_2)_{[m]} = \text{type}(\mathcal{J}_2)_m - \frac{1}{2}\dim(G) - \frac{1}{2}\dim(H) + 2\dim(\mathfrak{h}_M \cap \pi(L_2))_m.$$

Proof. As before, we begin by assuming that $a = 0$.

By restricting to a neighborhood of $f^{-1}(0)$, we may assume that G acts freely. Since the generalized Kähler structure is invariant under the group action, the canonical Riemannian metric k on M is also invariant. This induces a canonical connection $\theta \in \Omega^1(M, \mathfrak{g})$. By part (d) of Example 3.7, after applying an exact $B = d(h, \theta)$ transform, we may assume that $h = 0$.

As in the proof of Proposition 3.8, by the definition of generalized moment map $\mathfrak{g}_M \oplus df$ is a $\mathcal{J}_1$ invariant subbundle, and $(\mathfrak{g}_M \oplus df)|_{f^{-1}(0)}$ is isotropic. Define $\widehat{W} = (\mathfrak{g}_M \oplus df)^\perp \cap K(\mathfrak{g}_M \oplus df)^\perp \subset TM \oplus T^*M$, and let $\widehat{\mathcal{J}_1}$ and $\widehat{\mathcal{J}_2}$ be the restriction of $\mathcal{J}_1$ and $\mathcal{J}_2$ to $\widehat{W}$. Let $(\widetilde{\mathcal{J}_1}, \widetilde{\mathcal{J}_2})$ be generalized almost Kähler structure on M_0 induced by $\widehat{\mathcal{J}_1}$ and $\widehat{\mathcal{J}_2}$ under the restriction to $f^{-1}(0)$, isomorphism from $\widehat{W}|_{f^{-1}(0)}$ to $(\mathfrak{g}_M \oplus df)|^\perp_{f^{-1}(0)}/(\mathfrak{g}_M \oplus df)|_{f^{-1}(0)}$, and the quotient map from $f^{-1}(0)$ to M_0.

In Lemma 3.8, we checked that $\widetilde{\mathcal{J}_1}$ is a generalized complex structure.

Let $L_2 \subset TM_{\mathbb{C}} \oplus T^*_{\mathbb{C}}M$, $\widehat{L}_2 \subset \widehat{W}_{\mathbb{C}} = \widehat{W} \otimes \mathbb{C}$, and $\widetilde{L}_2 \subset T_{\mathbb{C}}M_0 \oplus T^*_{\mathbb{C}}M_0$ be $\sqrt{-1}$ eigenbundles of $\mathcal{J}_2$, $\widehat{\mathcal{J}_2}$, and $\widetilde{\mathcal{J}_2}$, respectively. Since $\mathcal{J}_2$ is a generalized complex structure, L_2 is closed under the Courant bracket. Since $K = -\mathcal{J}_1\mathcal{J}_2$, and $\mathcal{J}_1(\mathfrak{g}_M) = df$, $K(\mathfrak{g}_M) = \mathcal{J}_2(df)$ and $K(df) = \mathcal{J}_2(\mathfrak{g}_M)$. Therefore $\widehat{L}_2 = L_2 \cap \widehat{W}_{\mathbb{C}} = L_2 \cap \mathfrak{g}_M^\perp \cap df^\perp$. Therefore, by Lemmas 2.1 and 2.2, $\widehat{L}_2$ is also closed under the Courant bracket. Moreover, since $\widetilde{L}_2$ is the image of $\widehat{L}_2$ under the natural restriction and quotient maps, it is also closed under Courant bracket by the same lemmas. Therefore, $(\widetilde{\mathcal{J}_1}, \widetilde{\mathcal{J}_2})$ is a generalized Kähler structure.

The formulas on types follow directly from Lemma 4.2.

This proves the case $a = 0$. For $a \neq 0$, as in Lemma 3.8 let ω be the Kirillov-Kostant symplectic form on $\mathcal{O}{-a}$, and let J be the natural invariant complex structure which is compatible with ω. Let $(\mathcal{J}_\omega, \mathcal{J}_J)$ be the induced generalized Kähler structure on $\mathcal{O}_a$; it has type $(0, \frac{1}{2}\dim(\mathcal{O}_{-a}))$ and inclusion is a generalized moment map for the co-adjoint G action on $\mathcal{O}_{-a}$. Hence $(M \times O_{-a}, \mathcal{J}_1 \times \mathcal{J}_\omega, \mathcal{J}_2 \times \mathcal{J}_J)$ is a generalized Kähler manifold, $\text{type}(\mathcal{J}_1 \times \mathcal{J}_\omega)_{(m,b)} = \text{type}(\mathcal{J}_1)_m$ and $\text{type}(\mathcal{J}_2 \times \mathcal{J}_J)_{(m,b)} = \text{type}(\mathcal{J}_2)_m + \frac{1}{2}\dim(O_a)$ for all $m \in M$, and $\mu_a(x, \nu) = \mu(x) - \nu$ is a generalized moment map for the diagonal action of G on $M \times \mathcal{O}_{-a}$. Finally, it is easy to check that the intersection of $\mathfrak{g}_{M \times O_{-a}}$ with the projection of the $\sqrt{-1}$ eigenbundle of $\mathcal{J}_2 \times \mathcal{J}_J$ to $T_{\mathbb{C}}(M \times O_{-a})$ is isomorphic to $\mathfrak{h}_M \cap \pi(L_2)$. Since M_a can be identified with $\mu_a^{-1}(0)/G$, the result follows from case that $a = 0$. □

Example 4.7. The generalized Kähler quotient of the Hamiltonian generalized Kähler manifold associated to a Hamiltonian Kähler manifold is the generalized Kähler manifold associated to the Kähler quotient.

See Example 3.11 to understand how transforming the generalized Kähler structure transforms the quotient structure.

Example 4.8. Let (M, g, I, J, K) be a hyper-Kähler structure, and let ω_I, ω_J and ω_K be the Kähler two forms that correspond to the complex structure I, J and K respectively. As shown in [Gua04], we can construct a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ as follows:

$$\mathcal{J}_1 = \begin{pmatrix} 1 & 0 \\ \omega_K & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}(\omega_I^{-1} - \omega_J^{-1}) \\ \omega_I - \omega_J & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\omega_K & 1 \end{pmatrix}, \tag{4.3}$$

$$\mathcal{J}_2 = \begin{pmatrix} 1 & 0 \\ -\omega_K & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}(\omega_I^{-1} + \omega_J^{-1}) \\ \omega_I + \omega_J & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \omega_K & 1 \end{pmatrix}. \tag{4.4}$$

By definition, both $\mathcal{J}_1$ and $\mathcal{J}_2$ are of type zero. Suppose there is a Hamiltonian G-action on (M, g, I, J, K) with the equivariant moment map $\mu = (\mu_I, \mu_K, \mu_J) : M \to \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*$, i.e., for any $\xi \in \mathfrak{g}$, we have $\iota_{\xi_M}\omega_I = d\mu_I^\xi$, $\iota_{\xi_M}\omega_J = d\mu_J^\xi$, and $\iota_{\xi_M}\omega_K = d\mu_K^\xi$. Set $f = \mu_I - \mu_J : M \to \mathfrak{g}^*$. Then

(4.5)

$$\begin{aligned}
\mathcal{J}_1 df^\xi &= \begin{pmatrix} 1 & 0 \\ \omega_K & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}(\omega_I^{-1} - \omega_J^{-1}) \\ \omega_I - \omega_J & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\omega_K & 1 \end{pmatrix} \begin{pmatrix} 0 \\ df^\xi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \omega_K & 1 \end{pmatrix} \begin{pmatrix} 0 & -\frac{1}{2}(\omega_I^{-1} - \omega_J^{-1}) \\ \omega_I - \omega_J & 0 \end{pmatrix} \begin{pmatrix} 0 \\ df^\xi \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ \omega_K & 1 \end{pmatrix} \begin{pmatrix} -\xi \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\xi \\ -d\mu_K^\xi \end{pmatrix}
\end{aligned}$$

Thus $f + \sqrt{-1}\,\mu_K$ is a generalized moment map for the G action on the generalized Kähler manifold (M, J_1, J_2). Let θ be the canonical connection one form, let $B = d(\mu_K\theta)$, and let $\mathcal{J}_i'$ be the B-transform of $\mathcal{J}_i$. Then $(\mathcal{J}_1', \mathcal{J}_2)$ is a generalized Kähler structure which satisfies $\mathcal{J}_1' df^\xi = \xi$ for any $\xi \in \mathfrak{g}$ in a tubular neighborhood of the level set $f^{-1}(0)$. Assume that G acts freely on $f^{-1}(0)$. Proposition 4.6 then asserts that there is a reduced generalized Kähler structure $(\widetilde{J}_1, \widetilde{J}_2)$ on the quotient $M_0 = f^{-1}(0)/G$. Both J_1 and J_2 are of type zero at every point, G is the stabilizer of 0, and $\pi(L_2) = T_{\mathbb{C}}(M)$. Hence, by the type formula in Proposition 4.6 $\widetilde{J}_1$ is of type zero, whereas $\widetilde{J}_2$ is of type $\dim(G)$.

Assume in addition that 0 is a regular value for the map

$$\mu = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \oplus \mathfrak{g}^* \oplus \mathfrak{g}^*.$$

Then the hyper-Kähler quotient $S := \left(\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)\right)/G$ is submanifold sitting inside M_0 of codimension $2\dim G$. We have a natural inclusion map $i : S \to M_0$.

Let $(\widetilde{J}_1^S, \widetilde{J}_2^S)$ be the generalized Kähler structure on S induced by the quotient hyper-Kähler structure on S. Let $\widetilde{L}_i^S$ be the $\sqrt{-1}$ eigenbundle of $\widetilde{J}_i^S$, and let $\widetilde{L}_i$ be the $\sqrt{-1}$ eigenbundle of $\widetilde{J}_i$, $i = 1, 2$. Since the restriction of B to $\mu_I^{-1}(0) \cap \mu_J^{-1}(0) \cap \mu_K^{-1}(0)$ vanishes, we have that $\widetilde{J}_i^S = \{X + i^*\alpha \mid X + \alpha \in \widetilde{L}_i \cap (T_{\mathbb{C}}S \oplus T^*_{\mathbb{C}}M_0)\}$, $i = 1, 2$. The submanifolds of a generalized complex manifold is studied extensively in [BB03]. Using their terminology, we see that $(S, \widetilde{J}_i^S)$ is exactly a generalized complex submanifold of $(M_0, \widetilde{J}_i)$.

5. Constructing bi-Hermitian structures

In this section we are going to present a simple explicit construction of bi-Hermitian structure on $\mathbb{CP}^N$, Hirzebruch surfaces, $\mathbb{CP}^2$ blown up at an arbitrary number of points, and complex Grassmannians.

We will do this by constructing non-standard generalized Kähler structures on these spaces. Since each of these manifolds can be expressed as a symplectic quotient of $\mathbb{C}^n$, we start with the standard Kähler structure on $\mathbb{C}^n$. Using the deformation theory for generalized complex structures developed in [Gua04], we deform this to another invariant generalized Kähler structure. These techniques are particularly easy and explicit in this very simple example; we do not need to resort to any global analysis. Then we use the quotient construction we developed in Section 4 to construct a generalized Kähler structure on the quotient space which is not the B-transform of a genuine Kähler structure, although in each case the first generalized complex structure is the one induced from the standard symplectic structure. By the connection between generalized Kähler structures and bi-Hermitian structures which was established by Gualtieri [Gua04], and which we explain below, this induces a bi-Hermitian structure on each manifold.

As an aside, we will show that there exist a strongly bi-Hermitian structure on the generalized Kähler quotient space we discussed in Example 4.8. This provides a class of new examples of strongly bi-Hermitian manifolds.

5.1. **Review.** We begin with a brief review; all the material in this subsection, with the exception of material specifically attributed to other authors, was taken from [Gua04].

Definition 5.1. *(*[AGG99]*) A* **bi-Hermitian structure** *on a manifold* M *is a triple* (g, J_+, J_-)*, where* g *is a Riemannian metric and* J_+ *and* J_- *are complex structures which are orthogonal (with respect to* g*), induce the same orientation, and satisfy* $J_+(x) \neq \pm J_-(x)$ *for some* $x \in M$*. A* **strongly bi-Hermitian structure** *is a bi-Hermitian structure so that* $J_+(x) \neq \pm J_-(x)$ *for all* $x \in M$.

Given a generalized Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$, let $K = -\mathcal{J}_1\mathcal{J}_2$ be the associated positive definite metric. Recall that $K^2 = 1$, and let C_+ denote the $+1$ eigenspace of K. Since $C_+ \subset TM \oplus T^*M$ is positive definite and T^*M is isotropic, the natural projection $\pi\colon C_+ \to TM$ is an isomorphism.

Therefore, $\langle\cdot,\cdot\rangle$ descends to a Riemannian metric k on M. Since $\mathcal{J}_1$ and $\mathcal{J}_2$ commute with K, they both preserve C_+. Therefore, $\mathcal{J}_1$ and $\mathcal{J}_2$ descend to almost complex structures J_+ and J_- on M which are orthogonal (with respect to k).

The following proposition combines Proposition 6.15 with Remarks 6.13 and 6.14 in [Gua04].

Proposition 5.2. [Gua04] *Given a generalized Kähler manifold $(M,\mathcal{J}_1,\mathcal{J}_2)$, the above construction defines orthogonal complex structures J_+ and J_- on M. If at least one of the $\mathcal{J}_i$ has even type, J_+ and J_- induce the same orientation. (After possibly replacing J_- by $-J_-$.) Finally, for any $x\in M$, $J_-(x)\neq \pm J_+(x)$ exactly if*

$$\{\mathrm{type}(\mathcal{J}_1)_x,\mathrm{type}(\mathcal{J}_2)_x\}\neq\{0,\frac{1}{2}\dim(M)\}.$$

Note that, in fact, if $\dim M = 4k$, then either both $\mathcal{J}_1$ and $\mathcal{J}_2$ have odd type, or they both have even type. In contrast, if $\dim M = 4k+2$, then one must always have odd type whereas the other has even type; therefore, the first condition in the above proposition is empty.

Let $\mathcal{J}$ be a generalized complex structure on a vector space V. Let $L\subset V_{\mathbb{C}}\oplus V_{\mathbb{C}}^*$ be the $\sqrt{-1}$ eigenspace of $\mathcal{J}$. Since L is maximal isotropic and $L\cap\overline{L}=\{0\}$, we can (and will) use the metric to identify L^* with $\overline{L}$.

Given $\epsilon\in\wedge^2L^*$, define

$$L_\epsilon=\{Y+\iota_Y\epsilon \mid Y\in L\}.$$

Then L_ϵ is maximal isotropic, and $L_\epsilon\cap\overline{L_\epsilon}=\{0\}$ if and only if the endomorphism

$$A_\epsilon=\begin{pmatrix}1 & \bar{\epsilon}\\ \epsilon & 1\end{pmatrix} : L\oplus\overline{L}\to L\oplus\overline{L} \tag{5.1}$$

is invertible. If it is invertible, there exists a unique generalized complex structure $\mathcal{J}_\epsilon$ on V whose $\sqrt{-1}$ eigenspace is L_ϵ. Note that A_ϵ is always invertible for ϵ sufficiently small.

Now let $(\mathcal{J}_1,\mathcal{J}_2)$ be a generalized Kähler structure on V. Let L_1 and L_2 denote the $\sqrt{-1}$ eigenspaces of $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. Then $L_1 = (L_1\cap L_2)\oplus\left(L_1\cap\overline{L_2}\right)$ and $L_2=(L_1\cap L_2)\oplus\left(\overline{L_1}\cap L_2\right)$. Thus $\epsilon\in C^\infty(\wedge^2\overline{L_2})$ fixes $\mathcal{J}_1$ if and only if ϵ takes $L_1\cap L_2$ to $L_1\cap\overline{L_2}$, i.e., if and only if ϵ is an element of $C^\infty\left((\overline{L_1}\cap\overline{L_2})\otimes(L_1\cap\overline{L_2})\right)$.

We are now ready to state the condition for L_ϵ to be closed under the Courant bracket, as proved in [LWP97], following the presentation in [Gua04].

We begin with two definitions. Although both can be defined more generally for any Lie algebroid, we will only state them for the case which interests us.

Definition 5.3. *Let $L\subset T_{\mathbb{C}}M\oplus T_{\mathbb{C}}^*M$ be an isotropic subbundle which is closed under the Courant bracket and let $\pi\colon L\to T_{\mathbb{C}}M$ denote the natural projection. The*

Schouten bracket *is the* $\mathbb{R}$*-bilinear map*

$$[\cdot,\cdot]\colon C^\infty(\wedge^p L)\times C^\infty(\wedge^q L)\to C^\infty(\wedge^{p+q-1}L)$$

which is characterized by the following two formulas:

$$[X_1\wedge\cdots\wedge X_p, Y_1\wedge\cdots\wedge Y_q]=\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\cdots\wedge\widehat{X}_i\wedge\cdots\wedge\widehat{Y}_j\wedge\cdots\wedge Y_q$$

for any X_i *and* Y_j *in* $C^\infty(L)$*, and*

$$[Y,f]=-[f,Y]=\pi(Y)f$$

for any $Y\in C^\infty(L)$ *and* $f\in C^\infty(M)$.

Definition 5.4. *Let* $L\subset T_{\mathbb{C}}M\oplus T^*_{\mathbb{C}}M$ *be an isotropic subbundle which is closed under the Courant bracket and let* $\pi\colon L\to T_{\mathbb{C}}M$ *denote the natural projection. The* **Lie algebroid derivative** *is a first order linear differential operator from* $C^\infty(\wedge^* L)$ *to* $C^\infty(\wedge^{*+1}L)$ *defined by*

$$\begin{aligned} d_L\sigma(X_0,\cdots,X_k) &= \sum_i(-1)^i\pi(X_i)\sigma(X_0,\cdots,\widehat{X}_i,\cdots,X_k)\\ &\quad+\sum_{i<j}(-1)^{i+j}\sigma([X_i,X_j],X_0,\cdots,\widehat{X}_i,\cdots,\widehat{X}_j,\cdots,X_k), \end{aligned} \tag{5.2}$$

where $\sigma\in C^\infty(\wedge^k L^*)$ *and* $X_i\in C^\infty(L)$.

Example 5.5. If L is the $\sqrt{-1}$ eigenspace of the generalized complex structure $\mathcal{J}_J$ associated to a complex structure J, then d_L is $\overline{\partial}$.

We will need the following special case of the theorem from [LWP97].

Theorem 5.6. *Let* $L\subset T_{\mathbb{C}}M\oplus T^*_{\mathbb{C}}M$ *be a maximal isotropic subbundle so that* $L\cap\overline{L}=\{0\}$ *which is closed under the Courant bracket. For any* $\epsilon\in\wedge^2\overline{L}$,

$$L_\epsilon=\{Y+\iota_Y\epsilon\mid Y\in L\}$$

is closed under Courant bracket if and only if ϵ *satisfies the Maurer-Cartan equation:*

$$d_L\epsilon+\frac{1}{2}[\epsilon,\epsilon]=0.$$

5.2. **Examples.** Now we are ready to turn to specific examples.

Example 5.7. Structures on $\mathbb{C}^n$

We will begin by deforming the generalized Kähler structure $(\mathcal{J}_\omega,\mathcal{J}_J)$ on $\mathbb{C}^n$ which is induced by the standard genuine Kähler structure (ω, J). (See Example 4.3). Note that while our ideas for deforming this structure are taken entirely from [Gua04], and we use many of his observations, the deformation is much easier and more explicit in this very simple case than in general. In particular, while we use many observations from that paper, our construction does not rely on any of the deeper theorems.

Since

$$\overline{L_1}\cap\overline{L_2} = \{Y - \sqrt{-1}\,\iota_Y\omega \mid Y \in T_{1,0}(M)\} \text{ and } L_1\cap\overline{L_2} = \{Z + \sqrt{-1}\,\iota_Z\omega \mid Z \in T_{1,0}(M)\},$$

for any global sections Y and Z of $T_{1,0}(M)$

$$\epsilon = Y \wedge Z + \iota_Y\omega \wedge \iota_Z\omega$$

$$= \frac{1}{2}(Y - \sqrt{-1}\,\iota_Y\omega)\wedge(Z + \sqrt{-1}\,\iota_Z\omega) - \frac{1}{2}(Z - \sqrt{-1}\,\iota_Z\omega)\wedge(Y + \sqrt{-1}\,\iota_Y\omega)$$

lies in $\mathbb{C}^\infty((\overline{L_1}\cap\overline{L_2})\otimes(L_1\cap\overline{L_2}))$.

If we restrict to any open bounded subset U of $\mathbb{C}^n$, then after multiplying ϵ by a sufficiently small positive number, A_ϵ will be invertible. Thus ϵ deforms $\mathcal{J}_J$ to a new generalized complex structure $\mathcal{J}_\epsilon$ on U while keeping $\mathcal{J}_\omega$ fixed. Moreover, $\pi(L_\epsilon)$ is spanned by $T_{1,0}\mathbb{C}^n$, Y, and Z. Thus, $\text{type}(\mathcal{J}_\epsilon)_z = n-2$ wherever $Y\wedge Z \neq 0$, and n at every other point.

Finally, the following lemma gives a simple condition which guarantees that L_ϵ is closed under the Courant bracket, and hence that $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is a generalized Kähler structure.

Lemma 5.8. *Assume that there exists a subset* $I \subset (1,\dots,n)$ *so that*

$$\epsilon = \sum_{i,j\in I} F_{ij}(z)\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j} + \sum_{i,j\in I} F_{ij}(z)\,d\overline{z_i}\wedge d\overline{z_j}.$$

If F_{ij} *is holomorphic and* $\dfrac{\partial F_{ij}}{\partial z_k} = 0$ *for all* i, j *and* $k \in I$, *then* L_ϵ *is closed under the Courant bracket.*

Proof. Since the Lie algebroid derivative d_{L_J} is $\overline{\partial}$ and $F(z)$ is holomorphic, $d_{L_J}\epsilon = 0$. Hence, by Theorem 5.6, the deformed generalized almost Kähler structure will be a generalized Kähler structure exactly if $[\epsilon,\epsilon] = 0$. This follows from the calculation below:

$$\left[F_{ij}(z)\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j}, F_{kl}(z)\frac{\partial}{\partial z_k}\wedge\frac{\partial}{\partial z_l}\right]$$

$$= F_{kl}\left(-\frac{\partial F_{ij}}{\partial z_k}\frac{\partial}{\partial z_l} + \frac{\partial F_{ij}}{\partial z_l}\frac{\partial}{\partial z_k}\right)\wedge\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j} + F_{ij}\left(-\frac{\partial F_{kl}}{\partial z_i}\frac{\partial}{\partial z_j} + \frac{\partial F_{kl}}{\partial z_j}\frac{\partial}{\partial z_i}\right)\wedge\frac{\partial}{\partial z_k}\wedge\frac{\partial}{\partial z_l} = 0.$$

Similarly,

$$\left[F_{ij}(z)\frac{\partial}{\partial z_i}\wedge\frac{\partial}{\partial z_j}, F_{kl}\,d\overline{z_k}\wedge d\overline{z_l}\right]$$

$$= F_{ij}\left(\frac{\partial F_{kl}}{\partial z_j}\frac{\partial}{\partial z_i} - \frac{\partial F_{kl}}{\partial z_i}\frac{\partial}{\partial z_j}\right)\wedge d\overline{z_k}\wedge d\overline{z_l} = 0$$

Finally,

$$[F_{ij}\,d\overline{z_i}\wedge d\overline{z_j}, F_{kl}\,d\overline{z_k}\wedge d\overline{z_l}] = 0.$$

□

Suppose that a compact Lie group G acts on $(\mathbb{C}^n, \omega, J)$ with proper moment map $\Phi\colon \mathbb{C}^n \to \mathfrak{g}^*$. Consider $\alpha \in \mathfrak{g}^*$ so that G acts freely on $\Phi^{-1}(\mathcal{O}_\alpha)$; let $M_\alpha = \Phi^{-1}(\mathcal{O}_\alpha)$ denote the symplectic quotient, and let $\mathfrak{h}$ denote the Lie algebra of the stabilizer H of α.

Assume that there exists a subset $I \subset (1, \dots, n)$ so that

$$\epsilon = \sum_{i,j \in I} F_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \sum_{i,j\in I} F_{ij}(z) d\overline{z_i} \wedge d\overline{z_j}.$$

Moreover, assume that F_{ij} is holomorphic and $\dfrac{\partial F_{ij}}{\partial z_k} = 0$ for all i, j and k in I. Since $\mathcal{O}_\alpha$ is bounded, by multiplying ϵ by a sufficiently small constant we may assume that A_ϵ is invertible on $\mathcal{O}_\alpha$. Then, applying Lemma 5.8, $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is an invariant generalized Kähler structure with generalized moment map Φ. Hence, by Proposition 4.6, there is a natural generalized Kähler Structure $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$ on the symplectic quotient M_α. Moreover, $\widetilde{\mathcal{J}}_\omega$ has type 0; in fact, it is the generalized complex structure associated to the usual symplectic structure on M_α. Hence, condition (ii) of Proposition 5.2 is automatically satisfied. So, by Proposition 5.2, $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$ will induce a bi-Hermitian structure on the reduced space as long as it is not the B-transform of a genuine Kähler structure. To check this, it is enough to check that $\text{type}(\widetilde{\mathcal{J}}_\epsilon)_{[z]} \neq \frac{1}{2} \dim M_\alpha$ for at least some $[z] \in M_\alpha$. Since $\text{type}(\mathcal{J}_\epsilon)_z < N$ for generic $z \in \mathbb{C}^n$, by Proposition 4.6, it is enough to check that $\mathfrak{h}_{\mathbb{C}^n} \cap \pi(L_\epsilon) = \{0\}$ at generic points.

Example 5.9. ($\mathbb{CP}^N$ **for** $N \geq 2$)

We now construct a bi-Hermitian structure on $\mathbb{CP}^N$ for $N \geq 2$.

Let S^1 act on $\mathbb{C}^{N+1}$ via

$$\lambda \cdot (z_0, \dots, z_N) = (\lambda z_0, \dots, \lambda z_N).$$

Note that this action preserves the Kähler structure (ω, J). Moreover,

$$\Phi(z) = \sum_i \frac{1}{2} |z_i|^2$$

is a moment map, S^1 acts freely on $\Phi^{-1}(1)$, and the reduced space $M_1 = \Phi^{-1}(1)/S^1$ is $\mathbb{CP}^N$.

Let

$$\epsilon = z_0^2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + z_0^2 d\bar{z}_1 \wedge d\bar{z}_2.$$

After multiplying ϵ by a sufficiently small positive constant, A_ϵ is invertible, so ϵ deforms $(\mathcal{J}_\omega, \mathcal{J}_J)$ to a new generalized almost Kähler structure $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ on $\mathbb{C}^N$, so that $\text{type}(\mathcal{J}_\epsilon)_z$ is $N+1$ if $z_0 = 0$ and is $N-1$ otherwise. Since z_0^2 is holomorphic and $\dfrac{\partial z_0^2}{\partial z_1} = \dfrac{\partial z_0^2}{\partial z_2} = 0$, by Lemma 5.8 $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is in fact a generalize Kähler structure.

Since ϵ is S^1 invariant, $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is also S^1 invariant. Hence, by Proposition 4.6, there is a natural generalized Kähler structure $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$ on the quotient space $\mathbb{CP}^N = \Phi^{-1}(1)/S^1$.

Note that the fundamental vector generated by the action is

$$X = \frac{\sqrt{-1}}{2} \sum_i \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) ,$$

and hence X does not lie in $\pi(L_\epsilon)$ at any point of $\mathbb{C}^{N+1}$, where L_ϵ is the $\sqrt{-1}$ eigenbundle of $\mathcal{J}_\epsilon$. It follows immediately from Proposition 4.6 that $\text{type}(\widetilde{\mathcal{J}_\omega})_{[z]} = 0$ for all $[z] \in \mathbb{CP}^N$, whereas $\text{type}(\widetilde{\mathcal{J}}_\epsilon)_{[z]} = N$ if $z_0 = 0$, otherwise $\text{type}(\widetilde{\mathcal{J}}_\epsilon)_{[z]} = N - 2$. By Proposition 5.2 $(\widetilde{\mathcal{J}}_\omega, \widetilde{\mathcal{J}}_\epsilon)$ gives us a bi-Hermitian structure on $\mathbb{CP}^N$.

In the case of $N = 2$, the above construction actually gives us a $SU(2)$-invariant bi-Hermitian structure. Note that the standard action of $SU(2)$ on C^2 can be extended to C^3 by letting $SU(2)$ act on the first component trivially. This $SU(2)$ action commutes with the standard S^1 action on C^3 and therefore descends to a $SU(2)$ action on $\mathbb{CP}^2$. Since both ω and ϵ are $SU(2)$-invariant, the deformed generalized Kähler pair $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is $SU(2)$-invariant as well. Since the $SU(2)$ action on C^3 commutes with the standard S^1 action, we conclude that the quotient generalized Kähler structure must be $SU(2)$-invariant.

Example 5.10. Toric varieties

We will now construct bi-Hermitian structures on many, but not all, toric varieties, including all Hirzebruch surfaces and the blow up of $\mathbb{CP}^N$ at arbitrarily many points.

Let an n dimensional torus T with Lie algebra $\mathfrak{t}$ act on a compact symplectic manifold (M, ω) with moment map $\Psi\colon : M \to \mathfrak{t}^*$. Let $\Delta \subset \mathfrak{t}^*$ be the moment polytope. Let $\eta_1, \dots, \eta_N \in \mathfrak{t}$ be the primitive outward normals to the facets of Δ. Define $p\colon \mathbb{R}^N \to \mathfrak{t}$ by $p(e_i) = \eta_i$. Let K be the kernel of the associated map from $(S^1)^N$ to $\mathfrak{t}$. Let K act on $\mathbb{C}^N$ via its inclusion into $(S^1)^N$; let $\Phi\colon \mathbb{C}^N \to \mathfrak{k}^*$ denote the resulting moment map. There exists some $\xi \in \mathfrak{k}^*$ so that K acts freely on $\Phi^{-1}(\xi)$ and M is equivariantly symplectomorphic to the reduced space

$$M_\xi = \Phi^{-1}(\xi)/K.$$

Now assume that there exists $\alpha \in \mathfrak{t}^*$ so that $\alpha(\eta_1) = \alpha(\eta_2) = -1$, but $\alpha(\eta_i) \geq 0$ for all other i. Note that this condition is not satisfied for all toric symplectic manifolds, even in two dimensions. For example, it is not satisfied for $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown up at the four fixed points. On the other hand, it is satisfied in many cases, including Hirzebruch surfaces and $\mathbb{CP}^N$ blown up at a sequence of points as long as one picks those points carefully, for example, blow up in a sequence of points so that each point lies on $[0, z_1, z_2]$.

Since $\alpha(\eta_i) \geq 0$ for all $i \geq 3$, we may define

$$\epsilon = \left(\prod_{i\geq 3} z_i^{\alpha(\eta_i)}\right) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} + \left(\prod_{i\geq 3} z_i^{\alpha(\eta_i)}\right) d\overline{z_1} \wedge d\overline{z_2}.$$

After multiplying ϵ by a sufficiently small positive constant, A_ϵ is invertible. Therefore ϵ deforms $(\mathcal{J}_\omega, \mathcal{J}_J)$ to a new generalized almost Kähler structure $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ on $\mathbb{C}^N$ so that $\text{type}(\mathcal{J}_\epsilon)_z = N - 2$ if $z_j \neq 0$ for all j such that $\alpha(\eta_j) > 0$, and otherwise is equal to N. Since $= \prod_{i\geq 3} z_i^{\alpha(\eta_i)}$ is holomorphic and annihilated by $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$, by Lemma 5.8, $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is a generalized Kähler structure. By construction, ϵ is K-invariant, so $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ descends to a generalized Kähler structure $(\widetilde{\mathcal{J}_\omega}, \widetilde{\mathcal{J}_\epsilon})$ on the reduced space M_ξ. Finally, for any $\beta \in \mathfrak{k}$, let β_i denote the i'th coordinate of its natural inclusion into $\mathbb{R}^N$. Then

$$\beta_{\mathbb{C}^N} = \sum_i \frac{\sqrt{-1}\,\beta_i}{2}\left(z_i\frac{\partial}{\partial z_i} - \overline{z}_i\frac{\partial}{\partial \overline{z}_i}\right).$$

Since our assumptions rule out $\eta_1 = -\eta_2$, $\beta_i \neq 0$ for some i which is not 1 or 2. Hence, $\beta_{\mathbb{C}^N} \notin \pi(L_\epsilon)$. Thus, we get a bi-Hermitian structure.

Example 5.11. Grassmannians

Consider the natural action of $G = U(n)$ on $M = \mathbb{C}^n \otimes \mathbb{C}^m$ with the moment map $\Phi \colon M \to \mathfrak{g}^*$ given by

$$\Phi(z) = \sum_{j=1}^{m} \begin{pmatrix} z_{j1}\overline{z_{j1}} & z_{j1}\overline{z_{j2}} & \cdots & z_{j1}\overline{z_{jn}} \\ z_{j2}\overline{z_{j1}} & z_{j2}\overline{z_{j2}} & \cdots & z_{j2}\overline{z_{jn}} \\ \cdots & \cdots & \cdots & \cdots \\ z_{jn}\overline{z_{j1}} & z_{jn}\overline{z_{j2}} & \cdots & z_{jn}\overline{z_{jn}} \end{pmatrix}. \tag{5.3}$$

Here, we have labeled the coordinates $z_{11}\ldots, z_{1n}, \ldots, z_{m1}, \ldots, z_{mn}$, and identified $\mathfrak{g}^*$ with $n \times n$ matrices A such that $A = \overline{A}^t$. Let $\mathcal{O}_I$ be the coadjoint orbit containing the identity matrix I. Note that G acts freely on $\Phi^{-1}(\mathcal{O}_I)$, and furthermore that the reduced space $\Phi^{-1}(\mathcal{O}_I)/G$ is the Grassmannian of n planes in $\mathbb{C}^m$.

Of course, $\sum_{i=1}^n z_{i1}\overline{z_{i1}}$ is $U(n)$ invariant. For the same reason, $\sum_{i=1}^n z_{i1}\, d\overline{z_{i2}}$ and $\sum_{i=1}^n z_{i1}\, d\overline{z_{i3}}$ are $U(n)$ invariant. Therefore,

$$\epsilon = \left(\sum_{i=1}^n z_{i1}\frac{\partial}{\partial z_{i2}}\right) \wedge \left(\sum_{i=1}^n z_{i1}\frac{\partial}{\partial z_{i3}}\right) + \left(\sum_{i=1}^n z_{i1}\, d\overline{z_{i2}}\right) \wedge \left(\sum_{i=1}^n z_{i1}\, d\overline{z_{i3}}\right)$$

is also $U(n)$ invariant.

After multiplying ϵ by a sufficiently small positive constant A_ϵ is invertible, so it deforms $(\mathcal{J}_\omega, \mathcal{J}_J)$ to a new generalized almost Kähler structure $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ so that $\text{type}(\mathcal{J}_\epsilon)_z = nm - 2$ unless $z_{i1} = 0$ for all i, in which case it is nm. By Lemma 5.8, $(\mathcal{J}_\omega, \mathcal{J}_\epsilon)$ is in fact a generalized Kähler structure.

Moreover, it is easy to see that $\mathfrak{g}_M \cap L_\epsilon = \{0\}$. Therefore, this gives rise to a bi-Hermitian structure on the Grassmannian.

Example 5.12. Strongly bi-Hermitian manifolds

We claim that the generalized Kähler quotient of the G-Hamiltonian generalized Kähler manifold associated to a G-Hamiltonian Hyper-Kähler manifold in Example 4.8 is always strongly bi-Hermitian unless $\dim(M) = 4\dim(G)$. (Note that in this case the hyper-Kähler quotient is a point.) By our discussion in Example 4.8, the two commuting generalized complex structures that define the quotient generalized Kähler structure are of type 0 and $\dim(G)$ respectively. Therefore, they define a strongly bi-Hermitian structure as long as the dimension of the quotient, which is $\dim(M) - 2\dim(G)$, is not $2\dim(G)$.

Appendix A. Twisted structures

It is easy to check that the results and methods in this paper carry over to the twisted case with only minor modifications. In particular, we can define twisted generalized complex and Kähler reduction.

We begin with a brief introduction, following [Gua04]. For a closed form $H \in \Omega^3(M)$, the **H-twisted Courant bracket** on $T_\mathbb{C}M \oplus T^*_\mathbb{C}M$ is given by

$$[X + \alpha, Y + \beta]_H = [X, Y] + L_X\beta - L_Y\alpha - \frac{1}{2}(d\iota_X\beta - d\iota_Y\alpha) + \iota_Y\iota_X H.$$

A **H-twisted generalized complex structure** on M is a generalized complex structure $\mathcal{J}$ so that the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$ is closed under the H-twisted Courant bracket. Similarly, a **H-twisted generalized Kähler structure** is a generalized almost Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ so that the $\sqrt{-1}$ eigenbundles of $\mathcal{J}_1$ and $\mathcal{J}_2$ are closed under the H-twisted Courant bracket. As the following example shows, this concept is most interesting when H represents a non-trivial cohomology class.

Example A.1. [Gua04] For any $B \in \Omega^2(M)$ and closed $H \in \Omega^3(M)$, the B-transform of a H-twisted generalized complex (or Kähler) structure is a $H + dB$-twisted generalized complex (or Kähler) structure.

It is easy to check that Lemma 2.1 and Lemma 2.2 still hold for the twisted Courant bracket, and that the same proofs work. More specifically, if $f : M \to \mathfrak{g}^*$ is a submersion, then $df^\perp_\mathbb{C} \subset T_\mathbb{C}M \oplus T^*_\mathbb{C}M$ is closed under the H-twisted Courant bracket for any closed $H \in \Omega^3(M)$. Therefore, if Γ is a sub-bundle of $T_\mathbb{C}M \oplus T^*_\mathbb{C}M$ which is closed under the H-twisted Courant bracket, then the image of $\Gamma \cap df^\perp_\mathbb{C}$ in $T_\mathbb{C}(f^{-1}(0)) \oplus T^*_\mathbb{C}(f^{-1}(0))$ is closed under the $H|_{f^{-1}(0)}$-twisted Courant bracket. Similarly, let G act freely on M and assume that $H \in \Omega^3(M)$ is closed and basic. Then H descends to a closed form $\widetilde{H} \in \Omega^3(M/G)$ and the set of G-invariant sections of $(\mathfrak{g}_M)^\perp_\mathbb{C}$ is closed under the H-twisted Courant bracket. Therefore, if Γ is a sub-bundle of $T_\mathbb{C}M \oplus T^*_\mathbb{C}M$ which is closed under the H-twisted Courant bracket, then the image of $\Gamma \cap (\mathfrak{g}_M)^{\perp}_{\mathbb{C}}$ in $T_{\mathbb{C}}(M/G) \oplus T^*_{\mathbb{C}}(M/G)$ is closed under the $\widetilde{H}$-twisted Courant bracket.

In this context, we will work with a slight variant of Definition 3.4.

Definition A.2. *Let a compact Lie group G with Lie algebra $\mathfrak{g}$ act on a manifold M, preserving a H-twisted generalized complex structure $\mathcal{J}$, where $H \in \Omega^3(M)^G$ is closed. Let $L \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ denote the $\sqrt{-1}$ eigenbundle of $\mathcal{J}$. A* **twisted generalized moment map** *is a smooth function $f \colon M \to \mathfrak{g}^*$ so that*

- *There exists a one form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$, called the* **moment one form**, *so that $\xi_M - \sqrt{-1}\,(df^\xi + \sqrt{-1}\,\alpha^\xi)$ lies in L for all $\xi \in \mathfrak{g}$, where ξ_M denote the induced vector field.*
- *f is equivariant.*

Example A.3.

a) Let G act on a generalized complex manifold with generalized moment map $f + \sqrt{-1}\,h$. Then f is a twisted generalized moment map with moment one form dh.

b) Let G act on an H-twisted generalized complex manifold $(M, \mathcal{J})$ with twisted generalized moment map f and moment one-form α. If $B \in \Omega^2(M)^G$, then G acts on the B-transform of $\mathcal{J}$ with twisted generalized moment map f and moment one form α', where $(\alpha')^\xi = \alpha^\xi + \iota_{\xi_M} B$ for all $\xi \in \mathfrak{g}$.

Let a compact Lie group G act on a twisted generalized complex manifold $(M, \mathcal{J})$ with twisted generalized moment map f. Let $\mathcal{O}_a$ be the co-adjoint orbit through $a \in \mathfrak{g}^*$. As before, if G acts freely on $f^{-1}(\mathcal{O}_a)$, then $\mathcal{O}_a$ consists of regular values and $M_a = f^{-1}(\mathcal{O}_a)/G$ is a manifold, which we still call the **generalized complex quotient**. The proof of Lemma 3.8 applies almost word for word to the following generalization.

Lemma A.4. *Let a compact Lie group G act on a H-twisted generalized complex manifold $(M, \mathcal{J})$ with a twisted generalized moment map $f \colon M \to \mathfrak{g}^*$. Let $\mathcal{O}_a$ be a co-adjoint orbit through $a \in \mathfrak{g}^*$. so that G acts freely on $f^{-1}(\mathcal{O}_a)$. Assume that the moment one-form is trivial and H is basic. Then H descends to $\widetilde{H} \in \Omega^3_0(M_a)$ and the generalized complex quotient M_a naturally inherits a $\widetilde{H}$-twisted generalized complex structure $\widetilde{\mathcal{J}}$.*

Moreover, for all $m \in f^{-1}(\mathcal{O}_a)$,

$$\text{type}(\widetilde{\mathcal{J}})_{[m]} = \text{type}(\mathcal{J})_m.$$

Example A.5.

a) Let a compact Lie group act on a (untwisted) generalized complex manifold with a real generalized moment map. Then Lemma 3.8 and Lemma A.4 yield identical generalized complex structures on M_a.

b) Let a compact Lie group G act on a H-twisted generalized complex manifold $(M, \mathcal{J})$ with twisted generalized moment map f :

$M \to \mathfrak{g}^*$. Assume that the moment one-form is trivial and H is basic. If $B \in \Omega^2(M)$ is basic, then it descends to $\widetilde{B} \in \Omega^2(M_a)$, the B-transform of $\mathcal{J}$ satisfies the same conditions, and its twisted generalized complex quotient is the $\widetilde{B}$-transform of the twisted generalized complex quotient of $\mathcal{J}$.

Lemma A.6. *Let a compact Lie group* G *act freely on a manifold* M*. Let* H *be an invariant closed three form and let* α *be an equivariant mapping from* $\mathfrak{g}$ *to* $\Omega^1(M)$*. Fix a connection* $\theta \in \Omega(M, \mathfrak{g})$*. Then if* $H + \alpha \in \Omega^3_G(M)$ *is equivariantly closed, there exists a natural form* $\Gamma \in \Omega^2(M)^G$ *so that* $\iota_\xi \Gamma = \alpha^\xi$*. Thus* $H + \alpha + d_G\Gamma \in \Omega^3(M)^G \subset \Omega^3_G(M)$ *is closed and basic and so descends to a closed form* $\widetilde{H} \in \Omega^3(M/G)$ *so that* $[\widetilde{H}]$ *is the image of* $[H + \alpha]$ *under the Kirwan map.*[2]

Proof. Since $H + \alpha$ is equivariantly closed

- $dH = 0$.
- $\iota_{\xi_M} H = d\alpha^\xi$ for all $\xi \in \mathfrak{g}$.
- $\iota_{\xi_M} \alpha^\eta = -\iota_{\eta_M} \alpha^\xi$ for all ξ and η in $\mathfrak{g}$.

Define $\beta \in \Omega^2(M)^G$ by $\beta(X, Y) = -\alpha^{\theta(X)}(\theta(Y)_M) = \alpha^{\theta(Y)}(\theta(X)_M)$ for every vector field X and Y. Define $\Gamma = \Gamma_\theta = -(\alpha, \theta) + \beta \in \Omega^2(M)^G$. Then $\iota_{\xi_M}\Gamma = \alpha^\xi$ for all $\xi \in \mathfrak{g}$. Since Γ is G-invariant, $\iota_{\xi_M} d\Gamma = -d\iota_{\xi_M}\Gamma = -d\alpha^\xi = -\iota_{\xi_M} H$. Therefore, $H + d\Gamma$ is basic. □

We can now prove the twisted version of Proposition 3.10.

Proposition A.7. *Let a compact Lie group* G *act on an* H*-twisted generalized complex manifold* $(M, \mathcal{J})$ *with twisted generalized moment map* $f : M \to \mathfrak{g}^*$ *and moment one-form* $\alpha \in \Omega^1(M, \mathfrak{g}^*)$*. Let* $\mathcal{O}_a$ *be a co-adjoint orbit through* $a \in \mathfrak{g}^*$ *so that* G *acts freely on* $f^{-1}(\mathcal{O}_a)$*. Assume that* $H + \alpha$ *is equivariantly closed. Given a connection on* $f^{-1}(\mathcal{O}_a)$*, the twisted generalized complex quotient* M_a *inherits a* $\widetilde{H}$*-twisted generalized complex structure* $\widetilde{\mathcal{J}}$*, where* $\widetilde{H}$ *is defined as in the Lemma above. Up to* B*-transform,* $\widetilde{J}$ *is independent of the choice of connection. Finally, for all* $m \in f^{-1}(\mathcal{O}_a)$*,*

$$\operatorname{type}(\widetilde{\mathcal{J}})_{[m]} = \operatorname{type}(\mathcal{J})_m.$$

Proof. By restricting to a neighborhood of $f^{-1}(\mathcal{O}_a)$, we may assume that G acts freely on M. Given a connection $\theta \in \Omega^1(M, \mathfrak{g})$, by the lemma above there exists $\Gamma \in \Omega^2(M)$ so that $\iota_{\xi_M}\Gamma = \alpha^\xi$. So the Γ-transform of $\mathcal{J}$ is a $H - d\Gamma$-twisted generalized complex structure with twisted generalized moment map f and trivial moment one-form, and hence induces a natural $\widetilde{H}$-twisted generalized complex structure $\widetilde{\mathcal{J}}$ on M_a by Lemma A.4.

Given another connection θ', then $\Gamma_\theta - \Gamma_{\theta'}$ is basic and hence descends to $\widetilde{\gamma} \in \Omega^2(M)$. Consequently, the resulting twisted generalized complex structure on M_a is the $\widetilde{\gamma}$ transform of $\widetilde{\mathcal{J}}$. □

[2] In fact, the analogous statement holds for any equivariantly closed form, but the proof in general is more involved.

Remark A.8. In particular, even if $H = 0$, if $[\alpha] \neq 0$ then in general $[\widetilde{H}]$ will also not vanish. Thus, in principal it may be possible to get non-trivially twisted quotients from non-twisted spaces.

Example A.9.

a) Let a compact Lie group act on a generalized complex manifold with generalized moment map $f + \sqrt{-1}\, h$. The generalized complex structure on M_a induced by Proposition A.7 is the $(h, d\theta)$-transform of the generalized complex structure induced by Proposition A.7. In particular, it may be twisted.

b) Let G act on a H-twisted generalized complex manifold $(M, \mathcal{J})$ with twisted generalized moment map f and moment one-form α. Assume that $H + \alpha$ is equivariantly closed. Fix a connection θ on $f^{-1}(\mathcal{O}_a)$. Given any $B \in \Omega^2(M)^G$, let $\Gamma \in \Omega^2(M)^G$ be the natural form associated to the closed form $d_G(B) \in \Omega^3_G(M)$ by Lemma A.6 . Then $B + \Gamma$ is basic and hence descends to a form $\widetilde{B} \in \Omega^2(M_a)$, and the twisted generalized complex quotient of the B-transform of $\mathcal{J}$ is the $\widetilde{B}$ transform of the twisted generalized complex quotient. Hence, we cannot get interesting new examples by applying B-transforms to the space upstairs.

A **twisted generalized moment map** and **moment one-form** for a group action on a twisted generalized Kähler manifold $(M, \mathcal{J}_1, \mathcal{J}_2)$ are simply a twisted generalized moment map and moment one-form for the twisted generalized complex structure $\mathcal{J}_1$. We are now ready to prove our final proposition.

Proposition A.10. *Let a compact connected Lie group* G *act on a* H*-twisted generalized Kähler manifold* $(M, \mathcal{J}_1, \mathcal{J}_2)$ *with twisted generalized moment map* $f\colon M \to \mathfrak{g}^*$ *and moment one-form* $\alpha \in \Omega^1(M, \mathfrak{g}^*)$*. Let* $\mathcal{O}_a$ *be a co-adjoint orbit through* $a \in \mathfrak{g}^*$ *so that* G *acts freely on* $f^{-1}(\mathcal{O}_a)$*. Assume* $H + \alpha \in \Omega^3_G(M)$ *is equivariantly closed. Then the generalized Kähler quotient* M_a *naturally inherits a* $\widetilde{H}$*-twisted generalized Kähler structure* $(\widetilde{\mathcal{J}}_1, \widetilde{\mathcal{J}}_2)$*, where* $\widetilde{H}$ *is defined as in Lemma A.6 using the canonical connection on* M*.*

Finally, let $\mathfrak{h}$ *be the Lie algebra of the stabilizer* H *of* a*, and let* L_2 *be the* $\sqrt{-1}$ *eigenbundle of* $\mathcal{J}_2$*. Then for all* $m \in M$*,*

$$\mathrm{type}(\widetilde{\mathcal{J}}_1)_{[m]} = \mathrm{type}(\mathcal{J}_1)_m, \qquad \textit{and}$$

$$\mathrm{type}(\widetilde{\mathcal{J}}_2)_{[m]} = \mathrm{type}(\mathcal{J}_2)_m - \frac{1}{2}\dim G - \frac{1}{2}\dim(H) + 2\dim(\mathfrak{h}_M \cap \pi(L_2))_m.$$

Proof. By restricting to a neighborhood of $f^{-1}(\mathcal{O}_a)$, we may assume that G acts freely on M. The Γ_θ transform of $(\mathcal{J}_1, \mathcal{J}_2)$ is a $H + d\Gamma_\theta$-twisted generalized Kähler structure with twisted generalized moment map f and trivial moment one-form. From this point, the proof of Proposition 4.6 goes through without change. □

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